# A State-Space Approach to a One-Dimensional Mathematical Model for the Dynamics of Phase Transitions in Pseudoelastic Materials

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*Submitted by Harlan W. Stech*

Received December 14, 1992

An abstract formulation of a general mathematical model for the dynamics of shape memory alloys is presented. Using a state-space approach, the model is written as a semilinear Cauchy problem in an appropriate Hilbert space. It is shown that the differential operator corresponding to the linear part of the system generates an exponentially stable analytic semigroup. For this semigroup, explicit decay rates are given in terms of the physical constants of the system and its continuity with respect to the model parameters is proved. A proof of local existence of solutions is given and open problems are discussed. 

## 1. INTRODUCTION

The development of smart materials and structures has captured considerable attention during the last few years. Due to their unique characteristics, shape memory alloys (SMAs) have been considered among the materials with potential for applications in this area. In particular, they are being tested as actuators and sensors in various control systems.

* The work of the author was supported in part by the Air Force Office of Scientific Research (AFOSR) under Grant F49620-92-J-0078 and by the Institute for Mathematics and Its Applications of the University of Minnesota with funds provided by the Office of Naval Research (ONR) through Grant N/N0014-93-1-0027 and by the National Science Foundation (NSF) through Grant NSF/DMS-9023978 while the author was a postdoctoral fellow at the IMA. Current address: Instituto de Desarrollo Tecnológico para la Industria Química, INTEC, Güemes 3450, Santa Fe 3000, Argentina.



0022-247X/95 $6.00

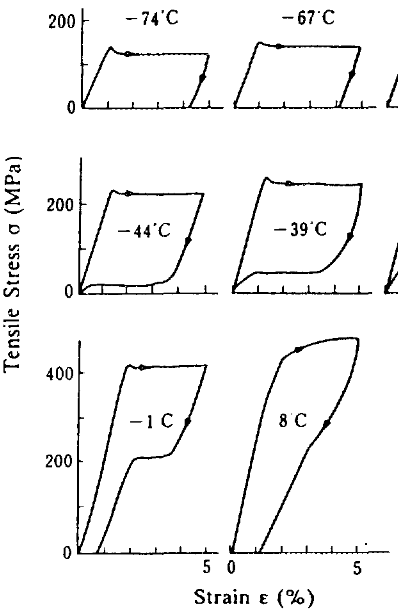


FIG. 1. Stress–strain curves obtained experimentally for Ti-51 at %Ni for different temperatures ([21]. Copyright © Gordon & Breach Science Publishers. Reprinted with permission.)

The name "shape memory alloys" comes from the fact that at low temperatures these intermetallic materials (chemical compounds of two or more elements) may sustain a residual deformation after the application of a large stress. However, their original shape can be completely restored simply by heating them above a certain critical temperature.

The behavior of these materials at low temperatures is elasto-plastic (Fig. 1, top). It is also called ferroelastic because of the similarity of the stress-strain relations with the field-magnetization curves of a ferromagnet. In this range of temperatures the load-deformation curves exhibit an elastic region at small loads, a plastic yield and a second elastic branch corresponding to large loads. This second elastic branch permits the body to withstand loads beyond the plastic yield, after which subsequent unloading leaves a residual deformation.

In the intermediate range of temperatures a plastic yield can still be observed (Fig. 1, center). Nevertheless, loading beyond the plastic yield followed by complete unloading does not lead to a residual deformation because of the existence of an intermediate elastic branch which the body reaches by creeping back after the load falls below a certain critical value.

This type of conduct is called pseudoelasticity and, for this reason, SMAs have also been termed pseudoelastic materials.

Finally, in the high-temperature range the behavior is almost linearly elastic for loads between certain bounds. However, application of large loads can produce a permanent deformation and the stress-strain curves may show a mixture of pseudoelasticity and shape recovery after unloading (Fig. 1, bottom). Some of the alloys which exhibit the above phenomena are AgCd, AuCd, CuAlNi, CuAuZn, CuSn, NiAl, and NiTi, to name only a few (see [21] for a complete list).

Although there is much to be studied and discovered in order to take full advantage of all their capabilities, SMAs have already found a broad variety of applications in aircraft, heat engines, orthodontic and other dental devices [4], robotic devices and actuators [21, 27], deployable antennas for spacecraft, pipe coupling devices, air conditioners, temperature switches and fuses [21], SMA hybrid composites [37], and in medicine as a substitute for the Harrington rod in the treatment of scoliosis [39] and as boneplates [9, 21].

The first observations of the shape memory effect (SME) go back to the 1930s. In 1938, Greninger of Harvard University and Mooradian of the Massachusetts Institute of Technology showed that temperature changes could produce and cause to disappear the martensite phase in brass. However, it was not until 1962, with the discovery of the Nitinol by Buehler [27], that rigorous in-depth studies were completed. Most of these initial efforts concentrated on metallurgical aspects and led to the publication of several books and papers related to the microscopic and mechanical properties [15, 16, 21, 36, 44]. Between 1968 and 1986 several mathematical models were proposed and studied [1–3, 18, 19, 26, 28–30, 46]). Most of these SMA models did not take into account the strong coupling between the thermal and the mechanical properties which characterize their behavior. Some of these static models dealt merely with the problem of finding a simple and appropriate fitting for the stress-strain relations that were able to describe and explain some of the properties observed experimentally. However, either temperature or stress was assumed constant everywhere, the processes were treated spatially pointwise, and time-dependent actions were excluded. These and other additional limitations make these models difficult to use in practical control design. In recent years, more complex mathematical models have been developed and studied in order to take into account thermal coupling, thermal memory, viscosity, and local curvature effects, as well as to allow time-dependent distributed and boundary inputs that could be used to control the system's behavior. These models often consist of a system of nonlinear coupled partial differential equations in displacement and temperature. The existence and uniqueness of solutions for these systems have been studied by several authors (see for instance [23, 24, 31, 33, 40, 41, 43], etc.).

The design, modeling, and control of intelligent systems and structures is an immensely complex problem and the subject of many ongoing interdisciplinary programs. There is a rapidly growing need for a unified theory in this area. This theory must be able to capture the unusual and complex characteristics of SMAs and at the same time it must be practical enough to allow for the development of computational algorithms for the design of controllers.

## 2. The Dynamic Equations: A Brief Review on the Existence and Uniqueness of Solutions

The conservation laws governing the thermomechanical processes in a one-dimensional shape-memory solid $\Omega$ give rise to the following system of partial differential equations (see [31] for details).

$$\rho u_{tt} - \beta\rho u_{xxt} - (\Psi_\varepsilon)_x + (\Psi_{\varepsilon_x})_{xx} = f, \qquad x \in \Omega,\ 0 \le t \le T \quad (2.1)$$

$$-\theta(\Psi_\theta)_t - k\theta_{xx} - \alpha k\theta_{xxt} - \beta\rho u_{xt}^2 = g, \qquad x \in \Omega,\ 0 \le t \le T. \quad (2.2)$$

The functions, variables, and parameters involved in (2.1), (2.2) have the following physical meaning: $u(x, t)$ = displacement, $\theta(x, t)$ = absolute temperature, $\rho$ = mass density, $k$ = thermal conductivity coefficient, $\beta$ = viscosity constant, $\alpha$ = thermal memory coefficient, $f(x, t)$ = distributed forces acting on the body, $g(x, t)$ = distributed heat sources, and $T$ = a certain prescribed final time. The function $\Psi$ represents the free energy density of the system and, in general, is assumed to be a function of the linearized shear strain $\varepsilon = u_x$, the spatial derivative of the strain $\varepsilon_x = u_{xx}$, and the temperature $\theta$. System (2.1), (2.2) must be complemented with relevant initial and boundary conditions and an explicit expression for the function $\Psi$.

Using a special type of Galerkin approximation, Niezgodka and Sprekels [31] first proved local existence of solutions to the system (2.1), (2.2) when the Hemholtz free energy density is given in the generalized Landau–Devonshire form $\Psi(\varepsilon, \theta) = \Psi_0(\theta) + \Psi_1(\theta)\varepsilon^2 + \Psi(\varepsilon)$ where $\Psi_0$, $\Psi_1$, $\Psi_2$ satisfy certain specific growth restrictions. The initial and boundary conditions in this case were taken as follows: $\theta(x, 0) = \theta_0(x)$, $u(x, 0) = u_0(x)$, and $u_t(x, 0) = u_1(x)$ for $x \in \Omega$, with $u_0 \in C^2(\overline{\Omega})$, $u_0|_{\partial\Omega} = u_0''|_{\partial\Omega} = 0$, $\|u_0\|_{H^3(\Omega)} \le E_0$, $u_1 \in H_0^1(\Omega)$, and $\theta_0 \in H^1(\Omega)$, $\theta_0(x) > \theta_s > 0$ for all $x \in \overline{\Omega}$, where $E_0$ and $\theta_s$ are two positive constants depending on the free energy, and $k(\partial\theta/\partial\nu) = k_1(\theta_\Gamma - \theta)$, $u \equiv 0$, on $\partial\Omega \times (0, T)$, where $\nu$ is the outward normal unit vector to $\partial\Omega$, $k_1$ is a positive constant and $\theta_\Gamma$ is the temperature of the surrounding medium. The uniqueness of solutions in this case was later proved by Hoffmann and Songmu [23, 24]. The first results on global existence are due to Dafermos and Hsiao [13, 14] and Niezgodka *et al.* [33]. The assumptions $\beta > 0$ (existence of viscous stress) and $\alpha > 0$ (existence of short thermal memory) played a very important role in all of the above mentioned articles. However, the physical meaning of the case $\alpha > 0$ is questionable since the second law of thermodynamics is not satisfied in this case, as can easily be proved by checking the Clausius–Duhem inequality for the entropy production.

The non-viscous case ($\beta = 0$) with no thermal memory ($\alpha = 0$) was treated later by Sprekels [43]. Here Sprekels used a Landau–Ginzburg potential of the form

$$\Psi(\varepsilon, \varepsilon_x, \theta) = \Psi_0(\theta) + \Psi_1(\theta)\Psi_2(\varepsilon) + \frac{\gamma}{2}\varepsilon_x^2, \tag{2.3}$$

where $\gamma$ is a positive constant. In this case the term $(\Psi_{\varepsilon_x})_{xx}$ in (2.1) takes the form $\gamma\varepsilon_{xxx} = \gamma u_{xxxx}$. This term provides sufficient smoothness for the existence of solutions. Even so, very strong growth restrictions on $\Psi$ were needed. One of these conditions, namely $|\Psi_1(\theta)| + |\Psi_1'(\theta)| + |\theta\Psi_1'(\theta)| \le C$ for all $\theta \ge 0$, excluded the physically relevant case in which $\Psi_1(\theta) = \alpha_2(\theta - \theta_1)$, $\Psi_2(\varepsilon) = \varepsilon^2$, $\Psi_3(\varepsilon) = -\alpha_4\varepsilon^4 + \alpha_6\varepsilon^6$, which corresponds to the so-called Landau–Devonshire potential, initially proposed and studied by Falk [18, 19], with the additional term $(\gamma/2)\varepsilon_x^2$.

In [40], Songmu studied the case $\beta = \alpha = 0$ and potentials of the form

$$\Psi(\varepsilon, \varepsilon_x, \theta) = \Psi_0(\theta) + \alpha_2(\theta - \theta_1)\varepsilon^2 - \alpha_4\varepsilon^4 + \alpha_6\varepsilon^6 + \frac{\gamma}{2}\varepsilon_x^2, \tag{2.4}$$

where $\alpha_2$, $\alpha_4$, $\alpha_6$, $\gamma$ are positive constants, $\theta_1$ is a critical temperature, and $\Psi_0$ is a certain smooth function of $\theta$, all depending on the material being considered. However, $-\Psi_0(\theta)$ was assumed to grow at least quadratically in $\theta$ which now excludes the physically relevant case

$$\Psi_0(\theta) = -C_v\theta \ln\left(\frac{\theta}{\theta_2}\right) + C_v\theta + C, \tag{2.5}$$

where $C_v$ is the specific heat, $\theta_2$ is a critical temperature, and $C$ is a constant representing a fixed reference energy level.

Finally, in 1988 Songmu and Sprekels [41] derived a priori estimates for the case $\beta = 0$, $\alpha = 0$, and Landau–Ginzburg potentials of the type

$$\Psi(\varepsilon, \varepsilon_x, \theta) = \Psi_0(\theta) + \alpha_1\theta\Psi_1(\varepsilon) + \Psi_2(\varepsilon) + \frac{\gamma}{2}\varepsilon_x^2 \tag{2.6}$$

where $\Psi_0$, $\Psi_1$, $\Psi_2$ satisfy certain weaker growth conditions than those imposed in [40] and [43]. These conditions are satisfied in particular by potentials of the form (2.4) with $\Psi_0$ as in (2.5). Convergent numerical approximations for this case were given by Niezgodka and Sprekels in [32].

In the next section we present a different approach to the study of this type of systems. Using a state-space framework, we transform the partial differential equations (2.1), (2.2) into an abstract semilinear Cauchy problem in an appropriate Hilbert space. We then prove that the operator $A(q)$ corresponding to the linear part of the system is the infinitesimal generator of an analytic semigroup $T(t; q)$ ($q$ being a vector containing the parameters of the system). This approach allows us not only to obtain explicit spectral decompositions for the operator $A(q)$ and the associated semigroup $T(t; q)$, but also to show that the semigroup $T(t; q)$ is exponentially stable with decay rate depending on the physical constants of the system. Finally, we show that the nonlinear part of the system is Lipschitz in the state variable in the topology of the graph of the associated differential operator, which leads to a proof of local existence of solutions. More importantly, this approach provides a friendly mathematical framework suitable for further developments. This framework is particularly appropriate for parameter identification and also for studying the asymptotic behavior of the system, problems which are still unsolved.

## 3. State-Space Formulation and Well-Posedness

Assume $\alpha = 0$ (no thermal memory), $\beta > 0$, $\Omega = (0, 1)$, and the free energy density given in a Landau–Ginzburg form like (2.4) with $\Psi_0(\theta)$ as in (2.5), i.e., we take

$$\begin{aligned} \Psi(\varepsilon, \varepsilon_x, \theta) = {} & -C_v\theta \ln\left(\frac{\theta}{\theta_2}\right) + C_v\theta + C + \alpha_2(\theta - \theta_1)\varepsilon^2 \\ & - \alpha_4\varepsilon^4 + \alpha_6\varepsilon^6 + \frac{\gamma}{2}\varepsilon_x^2. \end{aligned} \tag{3.1}$$

Under these assumptions, the system (2.1), (2.2) takes the form

$$\rho u_{tt} - \beta\rho u_{xxt} + \gamma u_{xxxx} = f(x, t) + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial \varepsilon}\Psi(u_x, u_{xx}, \theta)\right] \tag{3.2a}$$

$$C_v\theta_t - k\theta_{xx} = g(x, t) + 2\alpha_2\theta u_x u_{xt} + \beta\rho u_{xt}^2 \tag{3.2b}$$

for $x \in \Omega$, $0 \le t \le T$. We prescribe the initial conditions

$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ \theta(x, 0) = \theta_0(x), \qquad x \in \Omega, \tag{3.2c}$$

and for $0 \le t \le T$ the following boundary conditions

$$\begin{aligned} u(0, t) = u(1, t) = 0 = u_{xx}(0, t) = u_{xx}(1, t), \\ \theta_x(0, t) = 0, \qquad k\theta_x(1, t) = k_1(\theta_\Gamma(t) - \theta(1, t)). \end{aligned} \tag{3.2d}$$

where $\theta_\Gamma(t)$ is the temperature of the surrounding medium at time $t$ and $k_1$ is a positive coefficient. Next define $L(x, t) \doteq \theta_\Gamma(t)\cos(2\pi x)$ so that $L_x(0, t) = L_x(1, t) = 0$ and $L(1, t) = \theta_\Gamma(t)$ for all $t$, and let us make the transformation $\tilde{\theta}(x, t) \doteq \theta(x, t) - L(x, t)$. Then our IBVP takes the form

$$\rho u_{tt} - \beta\rho u_{xxt} + \gamma u_{xxxx} = f(x, t) + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial \varepsilon}\Psi(u_x, u_{xx}, \tilde{\theta} + L)\right] \tag{3.3a}$$

$$\begin{aligned} C_v\tilde{\theta}_t - k\tilde{\theta}_{xx} = g(x, t) + 2\alpha_2(\tilde{\theta} + L)u_x u_{xt} + \beta\rho u_{xt}^2 \\ - C_v\theta_\Gamma'(t)\cos(2\pi x) - 4k\pi^2\theta_\Gamma(t)\cos(2\pi x) \end{aligned} \tag{3.3b}$$

for $x \in \Omega$, $0 \le t \le T$,

$$\begin{aligned} u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \\ \tilde{\theta}(x, 0) = \theta_0(x) - \theta_\Gamma(0)\cos(2\pi x), \qquad x \in \Omega, \end{aligned} \tag{3.3c}$$

and

$$\begin{aligned} u(0, t) = u(1, t) = 0 = u_{xx}(0, t) = u_{xx}(1, t), \\ \tilde{\theta}_x(0, t) = 0,\ k\tilde{\theta}_x(1, t) + k_1\tilde{\theta}(1, t) = 0, \qquad 0 \le t \le T. \end{aligned} \tag{3.3d}$$

We assume that the functions $f(x, t)$ and $g(x, t)$ satisfy the following hypotheses:

(H1) There exist functions $K_g, K_f \in L_2(0, 1)$, $K_g(x) \ge 0$, $K_f(x) \ge 0$, such that $|f(x, t_1) - f(x, t_2)| \le K_f(x)|t_1 - t_2|$, and $|g(x, t_1) - g(x, t_2)| \le K_g(x)|t_1 - t_2|$ for all $x \in (0, 1)$, $t_1, t_2 \in [0, T]$.

(H2) $\theta_\Gamma \in H^1(0, T)$, $\theta_\Gamma$ and $\theta_\Gamma'$ are locally uniformly Lipschitz continuous, i.e., for each compact set $S \subset [0, T]$ there are constants $K_S, K_S' > 0$ such that $|\theta_\Gamma(t_1) - \theta_\Gamma(t_2)| \le K_S|t_1 - t_2|$, and $|\theta_\Gamma'(t_1) - \theta_\Gamma'(t_2)| \le K_S'|t_1 - t_2|$ for all $t_1, t_2 \in S$.

In order to formulate system (3.3.a)–(3.3.d) as a Cauchy problem in an abstract space, we define the state space $Z \doteq H_0^1(0, 1) \cap H^2(0, 1) \times L_2(0, 1) \times L_2(0, 1)$,

$$z(t) \doteq \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \doteq \begin{pmatrix} u(\cdot, t) \\ u_t(\cdot, t) \\ \tilde{\theta}(\cdot, t) \end{pmatrix} \in Z,$$

and the admissible parameter set $\mathcal{Q} \doteq \{q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \mid q \in \mathbb{R}_+^8\}$.

*Note.* We assume $k > 0$ and $k_1 > 0$ are known. Although this assumption is made mainly for reasons of simplicity, it is also rooted in the fact that heat conductivity is a physical parameter and can be estimated from laboratory experiments. In any case, only slight modifications are needed to consider the case in which $k$ and $k_1$ are also components of the parameter $q$.

We next define in $Z$ an inner product $\langle \cdot, \cdot \rangle_q$ depending on the parameter $q$ as follows,

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} \right\rangle_q \doteq \gamma \int_\Omega u'' \hat{u}'' + \rho \int_\Omega v \hat{v} + \frac{C_v}{k} \int_\Omega w \hat{w}, \tag{3.4}$$

and we denote by $Z_q$ the Hilbert space $Z$ with the inner product $\langle \cdot, \cdot \rangle_q$. The corresponding norm in $Z_q$ is denoted by $\|\cdot\|_q$. Then the IBVP (3.3.a)–(3.3.d) can be formally written as an abstract semilinear Cauchy problem in $Z_q$ as follows:

$$(\Sigma) : \begin{cases} \dot{z}(t) = A(q)z(t) + F(q, t, z(t)) & 0 \le t \le T \\ z(0) = z_0, \end{cases} \tag{3.5}$$

where

$$\operatorname{dom}(A(q)) \doteq \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q \;\middle|\; \begin{array}{l} u \in H^4(\Omega),\ u(0) = u(1) = 0 = u''(0) = u''(1), \\ v \in H_0^1(\Omega) \cap H^2(\Omega), \\ w \in H^2(\Omega),\ w'(0) = 0,\ kw'(1) = -k_1 w(1) \end{array} \right\} \tag{3.6}$$

and for

$$z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \operatorname{dom}(A(q)),$$

$$A(q)z = A(q)\begin{pmatrix} u \\ v \\ w \end{pmatrix} \doteq \begin{pmatrix} v \\ \beta v'' - \dfrac{\gamma}{\rho} u'''' \\ \dfrac{k}{C_v} w'' \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I & 0 \\ -\dfrac{\gamma}{\rho}\dfrac{\partial^4}{\partial x^4} & \beta \dfrac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \dfrac{k}{C_v}\dfrac{\partial^2}{\partial x^2} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{3.7}$$

and

$$z_0(\cdot) = \begin{pmatrix} u_0(\cdot) \\ u_1(\cdot) \\ \theta_0(\cdot) - \theta_\Gamma(0)\cos(2\pi\cdot) \end{pmatrix}.$$

Note that $\operatorname{dom}(A(q))$ is a subspace of $Z_q$ independent of $q \in \mathfrak{Q}$ since $k$ and $k_1$ are supposed to be known (i.e., they are not components of $q$).

The function $F(q, t, z) : \mathfrak{Q} \times \mathbb{R}_0^+ \times Z_q \to Z_q$ is defined by

$$F(q, t, z) = F\left(q, t, \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) \doteq \begin{pmatrix} 0 \\ f_2(q, t, z) \\ f_3(q, t, z) \end{pmatrix}, \tag{3.8}$$

where

$$f_2(q, t, z) \doteq \rho^{-1} f(\cdot, t) + \rho^{-1}\frac{\partial}{\partial x}[2\alpha_2(w + L(\cdot, t) - \theta_1)u' \\ -4\alpha_4 u'^3 + 6\alpha_6 u'^5], \tag{3.9a}$$

$$f_3(q, t, z) \doteq C_v^{-1} g(\cdot, t) + 2\alpha_2 C_v^{-1}(w + L(\cdot, t))u'v' + \beta\rho C_v^{-1}(v')^2 \\ -\theta_\Gamma'(t)\cos(2\pi\cdot) - 4k\pi^2 C_v^{-1} L(\cdot, t). \tag{3.9b}$$

Note that we have deliberately left on $f_2(q, t, z)$ the linear second order terms coming from $(\partial/\partial x)[(\partial/\partial\varepsilon)\Psi\,(u_x, u_{xx}, \tilde{\theta} + L)]$.

THEOREM 3.1. *For each $q \in \mathcal{Q}$ the operator $A(q) : \mathrm{dom}(A(q)) \subset Z_q \to Z_q$ as defined by* (3.6), (3.7) *is dissipative.*

*Proof.* Let

$$z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathrm{dom}(A(q))$$

so that $u \in H^4$, $u(0) = u(1) = 0 = u''(0) = u''(1)$, $v \in H_0^1 \cap H^2$, $w \in H^2$, $w'(0) = 0$, and $kw'(1) + k_1 w(1) = 0$. Integration by parts yields

$$\begin{aligned} \langle A(q)z, z\rangle_q &= \left\langle \begin{pmatrix} v \\ \beta v'' - \dfrac{\gamma}{\rho} u'''' \\ \dfrac{k}{C_v} w'' \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\rangle_q \\ &= -\rho\beta\|v'\|_{L_2}^2 - \|w'\|_{L_2}^2 - \frac{k_1}{k}[w(1)]^2 \\ &\le 0. \end{aligned}$$

Hence, $A(q)$ is dissipative.

THEOREM 3.2. *Let $q \in \mathcal{Q}$, and $Z_q$ and $A(q)$ be as above. Then the adjoint of $A(q)$, $A^*(q)$, is given by* dom $(A^*(q)) \doteq$ dom $(A(q))$, *and for*

$$z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathrm{dom}\,(A^*(q))$$

$$A^*(q)z = \begin{pmatrix} -v \\ \beta v'' + \dfrac{\gamma}{\rho} u'''' \\ \dfrac{k}{C_v} w'' \end{pmatrix} = \begin{pmatrix} 0 & -I & 0 \\ \dfrac{\gamma}{\rho}\dfrac{\partial^4}{\partial x^4} & \beta\dfrac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \dfrac{k}{C_v}\dfrac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{3.10}$$

*Proof.* We prove this theorem in two parts. First we show that dom

$(A(q)) \subset \operatorname{dom}(A^*(q))$ and $A^*(q)z$ is given by (3.10). In the second part we show that the opposite inclusion holds for the domains.

First Part: Let

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \operatorname{dom}\,(A(q)).$$

Then for any

$$\eta = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \operatorname{dom}\,(A(q))$$

we have

$$\begin{aligned}
\langle A(q)\eta, z\rangle_q &= \left\langle \begin{pmatrix} v \\ \beta v'' - \dfrac{\gamma}{\rho}\mu'''' \\ \dfrac{k}{C_v} w'' \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right\rangle_q \\
&= \gamma \int_\Omega v'' z_1'' + \rho \int_\Omega \left(\beta v'' - \frac{\gamma}{\rho} u''''\right) z_2 + \frac{C_v}{k} \int_\Omega \frac{k}{C_v} w'' z_3 \\
&= \gamma \int_\Omega v z_1'''' + \rho\beta \int_\Omega v z_2'' - \gamma \int_\Omega u'' z_2'' + \int_\Omega w z_3'',
\end{aligned}$$

where the last equality follows from integration by parts and the fact that $z_3(1)w'(1) = z_3'(1)w(1)$.

By rearranging terms it follows that

$$\begin{aligned}
\langle A(q)\eta, z\rangle_q &= \gamma \int_\Omega u''(-z_2'') + \rho \int_\Omega v \left(\beta z_2'' + \frac{\gamma}{\rho} z_1''''\right) + \frac{C_v}{k} \int_\Omega w \left(\frac{k}{C_v} z_3''\right) \\
&= \left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} -z_2 \\ \beta z_2'' + \dfrac{\gamma}{\rho} z_1'''' \\ \dfrac{k}{C_v} z_3'' \end{pmatrix} \right\rangle_q .
\end{aligned}$$

Hence if

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \operatorname{dom}(A(q))$$

then $z \in \operatorname{dom}(A^*(q))$ and

$$A^*(q)z = A^*(q)\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_2 \\ \beta z_2'' + \dfrac{\gamma}{\rho} z_1'''' \\ \dfrac{k}{C_v} z_3'' \end{pmatrix} = \begin{pmatrix} 0 & -I & 0 \\ \dfrac{\gamma}{\rho}\dfrac{\partial^4}{\partial x_4} & \beta\dfrac{\partial^2}{\partial x_2} & 0 \\ 0 & 0 & \dfrac{k}{C_v}\dfrac{\partial^2}{\partial x^2} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Second Part: Now let

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \operatorname{dom}(A^*(q)).$$

Then there exists

$$\tilde{z} = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{pmatrix} \in Z_q$$

such that for all

$$\eta = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \operatorname{dom}(A(q)),$$

$$\begin{aligned} 0 &= \langle A(q)\eta, z\rangle_q - \langle \eta, \tilde{z}\rangle_q \\ &= \left\langle \begin{pmatrix} v \\ \beta v'' - \dfrac{\gamma}{\rho} u'''' \\ \dfrac{k}{C_v} w'' \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{pmatrix} \right\rangle_q \end{aligned}$$

$$= \gamma \int_\Omega (v''z_1'' - u''\bar{z}_1'') + \rho \int_\Omega \left[ \left( \beta v'' - \frac{\gamma}{\rho} u'''' \right) z_2 - v\bar{z}_2 \right]$$

$$+ \frac{C_v}{k} \int_\Omega \left( \frac{k}{C_v} w'' z_3 - w\bar{z}_3 \right)$$

$$= \int_\Omega (\gamma v'' z_1'' + \rho\beta v'' z_2 - \rho v \bar{z}_2) - \gamma \int_\Omega (u''\bar{z}_1'' + u'''' z_2)$$

$$+ \frac{C_v}{k} \int_\Omega \left( \frac{k}{C_v} w'' z_3 - w\bar{z}_3 \right).$$

Since this equality must hold for all

$$\eta = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \operatorname{dom}(A(q)),$$

each one of the terms in the above expression must vanish, i.e.,

(a) $\int_\Omega \bar{z}_1'' u'' + z_2 u'''' = 0$ for all $u \in H^4$, $u|_{\partial\Omega} = u''|_{\partial\Omega} = 0$,

(b) $\int_\Omega (-\rho\bar{z}_2)v + (\gamma z_1'' + \rho\beta z_2)v'' = 0$ for all $v \in H_0^1 \cap H^2$,

(c) $\int_\Omega (-\bar{z}_3)w + \left( \frac{k}{C_v} z_3 \right) w'' = 0$ for all $w \in H^2$, $w'(0) = 0$, $kw'(1) + k_1 w(1) = 0$.

Now (a) implies that $\int_\Omega (\bar{z}_1'' h + z_2 h'') = 0$ for all $h \in H_0^1 \cap H^2$. In particular, this equality must hold for all $h \in H_0^2$. Then, by the Fundamental Lemma of the Calculus of Variations [17, pp. 31–32] there exist constants $a$ and $b$ such that

$$z_2(x) = ax + b - \int_0^x \int_0^s \bar{z}_1''(\xi)\, d\xi\, ds, \qquad \text{for } x \in \Omega.$$

Hence, $z_2 \in H^2$ and by differentiating twice the above expression we get

$$\bar{z}_1'' = -z_2'', \qquad z_2 \in H^2. \tag{3.11a}$$

Similarly, the Fundamental Lemma of the Calculus of Variations applied to (b) gives the existence of two constants $c$ and $d$ such that

$$\gamma z_1''(x) = -\rho\beta z_2(x) + cx + d + \rho \int_0^x \int_0^s \bar{z}_2(\xi)\, d\xi\, ds, \qquad \text{for } x \in \Omega.$$

Now, since $z_2 \in H^2$, the RHS in the above expression is in $H^2$. Hence $z_1 \in H^4$ and by differentiating twice we obtain

$$\tilde{z}_2 = \beta z_2'' + \frac{\gamma}{\rho} z_1'''', \qquad z_1 \in H^4. \tag{3.11b}$$

Finally, observe that (c) must hold in particular for all $w \in H_0^2$. Again, the Fundamental Lemma of the Calculus of Variations yields the existence of two constants $p$ and $q$ such that

$$\frac{k}{C_v} z_3(x) = px + q - \int_0^x \int_0^s (-\tilde{z}_3(\xi))\, d\xi\, ds, \qquad \text{for } x \in \Omega.$$

By differentiating twice the above expression we get

$$\tilde{z}_3 = \frac{k}{C_v} z_3'', \qquad z_3 \in H^2. \tag{3.11c}$$

Therefore, if

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \operatorname{dom}(A^*(q)),$$

then $z_1 \in H^4 \cap H_0^1$ (note $z_1 \in H_0^1$ because $z$ must belong to $Z_q$) and $z_2, z_3 \in H^2$. To show that dom $(A^*(q)) = \operatorname{dom}(A(q))$ it remains to show that $z_1''|_{\partial\Omega} = 0$, $z_2|_{\partial\Omega} = 0$, $z_3'(0) = 0$, and $kz_3'(1) + k_1 z_3(1) = 0$.

From (3.11a) and (a) we have that for all $u \in H_0^1 \cap H^4$ such that $u''|_{\partial\Omega} = 0$,

$$\begin{aligned} 0 &= \int_\Omega (-z_2''u'' + z_2 u'''') \\ &= -\left[u''z_2'|_{\partial\Omega} - \int_\Omega z_2'u'''\right] + \left[z_2 u'''|_{\partial\Omega} - \int_\Omega u'''z_2'\right] \\ &= z_2 u'''|_{\partial\Omega}. \end{aligned}$$

Since this must hold for all $u \in H_0^1 \cap H^4$ with $u''|_{\partial\Omega} = 0$, we must have

$$z_2|_{\partial\Omega} = 0. \tag{3.12}$$

From (3.11b) and (b) we get that for all $v \in H_0^1 \cap H^2$,

$$
\begin{aligned}
0 &= \int_\Omega \left[-\rho\left(\beta z_2'' + \frac{\gamma}{\rho} z_1''''\right) v + (\gamma z_1'' + \rho\beta z_2)v''\right] \\
&= -\rho\left[v\left(\beta z_2' + \frac{\gamma}{\rho} z_1'''\right)\Big|_{\partial\Omega} - \int_\Omega \left(\beta z_2' + \frac{\gamma}{\rho} z_1'''\right) v'\right] \\
&\quad + \left[(\gamma z_1'' + \rho\beta z_2)v'\Big|_{\partial\Omega} - \int_\Omega v'(\gamma z_1''' + \rho\beta z_2')\right] \\
&= (\gamma z_1'' + \rho\beta z_2)v'|_{\partial\Omega} = \gamma z_1'' v'|_{\partial\Omega}.
\end{aligned}
$$

The last equality follows from (3.12) since $z_2|_{\partial\Omega} = 0$. Since this must hold for all $v \in H_0^1 \cap H^2$, we must have $z_1''|_{\partial\Omega} = 0$.

Finally, from (3.11c) and (c) we get that for all $w \in H^2$ with $w'(0) = 0$ and $kw'(1) + k_1 w(1) = 0$ there holds

$$
\begin{aligned}
0 &= \int_\Omega (-z_3'' w + z_3 w'') \\
&= (-wz_3' + w'z_3)|_{\partial\Omega} \\
&= -[w(1)z_3'(1) - w(0)z_3'(0)] + w'(1)z_3(1) \\
&= -[w(1)z_3'(1) - w(0)z_3'(0)] - \frac{k_1}{k} w(1)z_3(1) \\
&= w(1)\left[-\frac{k_1}{k} z_3(1) - z_3'(1)\right] + w(0)z_3'(0).
\end{aligned}
$$

Therefore, we must have $-(k_1/k)z_3(1) - z_3'(1) = 0$ and $z_3'(0) = 0$, i.e., $z_3'(0) = 0$ and $kz_3'(1) + k_1 z_3(1) = 0$. Hence $z \in \operatorname{dom}(A(q))$ and therefore $\operatorname{dom}(A^*(q)) = \operatorname{com}(A(q))$. This completes the proof of Theorem 3.2. ∎

We shall now prove that $A(q)$ is the infinitesimal generator of an analytic semigroup. We shall achieve this in several steps.

Let $L_{2,\rho}(\Omega)$ denote the Hilbert space $L_2(\Omega)$ with inner product defined by $\langle u, v\rangle_{L_{2,\rho}} \doteq \rho \int_\Omega uv$ and let us define the operators $A_1(q)$ and $B_1(q)$ on $L_{2,\rho}(\Omega)$ by $\operatorname{dom}(A_1(q)) \doteq \{u \in H^4(\Omega) \mid u(0) = u(1) = u''(0) = u''(1) = 0\}$, $\operatorname{dom}(B_1(q)) \doteq H_0^1(\Omega) \cap H^2(\Omega)$, and

$$A_1(q)f \doteq \frac{\gamma}{\rho}\frac{\partial f}{\partial x^4}, \qquad f \in \operatorname{dom}(A_1(q)), \tag{3.13}$$

$$B_1(q)h \doteq -\beta\frac{\partial h}{\partial x^2}, \qquad h \in \operatorname{dom}(B_1(q)). \tag{3.14}$$

Note that both $\operatorname{dom}(A_1(q))$ and $\operatorname{dom}(B_1(q))$ are dense in $L_{2,\rho}(\Omega)$.

THEOREM 3.3. *Let $A_1(q)$: $\mathrm{dom}(A_1(q)) \subset L_{2,\rho}(\Omega) \to L_{2,\rho}(\Omega)$, and $B_1(q)$: $\mathrm{dom}(B_1(q)) \subset L_{2,\rho}(\Omega) \to L_{2,\rho}(\Omega)$ be as above. Then,* (i) *$A_1(q)$ is strictly positive and self-adjoint*; (ii) *$B_1(q)$ is strictly positive and self-adjoint, and* (iii) *$B_1(q) = (\beta\sqrt{\rho}/\sqrt{\gamma})\, A_1^{1/2}(q)$.*

*Proof.* (i) If $u \in \mathrm{dom}(A_1(q))$, then integration by parts yields

$$\langle A_1(q)u, u\rangle_{L_{2,\rho}} = \left\langle \frac{\gamma}{\rho} u'''', u\right\rangle_{L_{2,\rho}} = \gamma \int_\Omega u''''u = \gamma\|u''\|^2_{L_2(\Omega)} \geq 0.$$

Moreover, $\langle A_1(q)u, u\rangle_{L_{2,\rho}} = 0$ implies $\|u''\|_{L_2} = 0$ which in turn implies $u = 0$ since $u|_{\partial\Omega} = 0$. Hence, $A_1(q)$ is strictly positive. We shall prove now that $A_1(q)$ is self-adjoint. If $v \in \mathrm{dom}(A_1(q))$, then for any $u \in \mathrm{dom}(A_1(q))$ we have

$$\langle A_1(q)u, v\rangle_{L_{2,\rho}} = \gamma \int_\Omega u''''v = \gamma \int_\Omega v''''u = \rho \int_\Omega u \frac{\gamma}{\rho} v'' = \langle u, A_1(q)v\rangle_{L_{2,\rho}}.$$

Therefore, $v \in \mathrm{dom}(A_1^*(q))$ and $A_1^*(q)v = A_1(q)v$, i.e., $A_1(q)$ is symmetric. Now if $u \in \mathrm{dom}(A_1^*(q))$, then there exists $v \in L_{2,\rho}(\Omega)$ such that for all $w \in \mathrm{dom}(A_1(q))$,

$$0 = \langle A_1(q)w, u\rangle_{L_{2,\rho}} - \langle w, u\rangle_{L_{2,\rho}} = \rho \int_\Omega \left(\frac{\gamma}{\rho} w''''u - wv\right). \tag{3.15}$$

This equality must hold in particular for all $w \in H_0^4(\Omega)$. The Fundamental Lemma of the Calculus of Variations implies that there exist four constants $a$, $b$, $c$, and $d$ such that

$$\frac{\gamma}{\rho} u(x) = ax^3 + bx^2 + cx + d - \int_0^x \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} (-v(\xi))\, d\xi\, ds_3\, ds_2\, ds_1, \qquad \text{for } x \in \Omega.$$

Hence, $u \in H^4(\Omega)$ and, by differentiating four times, the above expression becomes $(\gamma/\rho)u'''' = v$. Substituting this expression into (3.15) we get

$$\begin{aligned} 0 &= \int_\Omega (w''''u - wu'''') = uw'''|_{\partial\Omega} - u'w''|_{\partial\Omega} - wu'''|_{\partial\Omega} + w'u''|_{\partial\Omega} \\ &= uw'''|_{\partial\Omega} + u''w'|_{\partial\Omega}. \end{aligned}$$

Since this equality must hold for all $w \in \mathrm{dom}(A_1(q))$, we conclude that $u|_{\partial\Omega} = u''|_{\partial\Omega} = 0$. Hence, $\mathrm{dom}(A_1^*(q)) = \mathrm{dom}(A_1(q))$ and $A_1(q)$ is self-adjoint.

(ii) If $u \in \mathrm{dom}(B_1(q))$, then $\langle B_1(q)u, u\rangle_{L_{2,\rho}} = \langle -\beta u'', u\rangle_{L_{2,\rho}} = -\beta\rho \int_\Omega u''u = \beta\rho\|u'\|_{L_2}^2 \geq 0$. Moreover, $\langle B_1(q)u, u\rangle_{L_{2,\rho}} = 0$ implies $\|u'\|_{L_2} = 0$ and, therefore, $u = 0$ since $u|_{\partial\Omega} = 0$. Thus, $B_1(q)$ is strictly positive. We will show now that $B_1(q)$ is self-adjoint. Let $v \in \mathrm{dom}(B_1(q))$. Then for any $u \in \mathrm{dom}(B_1(q))$ there holds $\langle B_1(q)u, v\rangle_{L_{2,\rho}} = \langle -\beta u'', v\rangle_{L_{2,\rho}} = -\beta\rho \int_\Omega u''v$. After integrating by parts twice and using $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, one obtains

$$\langle B_1(q)u, v\rangle_{L_{2,\rho}} = -\beta\rho \int_\Omega uv'' = \langle u, -\beta v''\rangle_{L_{2,\rho}} = \langle u, B_1(q)v\rangle_{L_{2,\rho}}.$$

Therefore, $v \in \mathrm{dom}(B_1^*(q))$ and $B_1^*(q)v = B_1(q)v$, i.e., $B_1(q)$ is symmetric.

Now if $u \in \mathrm{dom}(B_1^*(q))$, then there exists $v \in L_{2,\rho}(\Omega)$ such that for all $w \in \mathrm{dom}(B_1(q))$

$$0 = \langle B_1(q)w, u\rangle_{L_{2,\rho}} - \langle w, v\rangle_{L_{2,\rho}} = -\rho \int_\Omega (\beta w''u + wv). \tag{3.16}$$

This equality must hold for all $w \in H_0^2(\Omega)$. The Fundamental Lemma of the Calculus of Variations implies that there exist two constants $a$ and $b$ such that

$$\beta u(x) = ax + b - \int_0^x \int_0^s v(\xi)\, d\xi\, ds, \qquad \text{for } x \in \Omega.$$

Thus, $u \in H^2(\Omega)$ and, by differentiating twice, it follows that $\beta u'' = -v$. Substituting into (3.16) we get $0 = \int_\Omega (w''u - wu'') = uw'|_{\partial\Omega} - wu'|_{\partial\Omega} = uw'|_{\partial\Omega}$. Since this equality must hold for all $w \in \mathrm{dom}(B_1(q))$, we conclude that $u|_{\partial\Omega} = 0$. Hence, $\mathrm{dom}(B_1^*(q)) = \mathrm{dom}(B_1(q))$ and $B_1(q)$ is self-adjoint.

(iii) Since $A_1(q)$ is positive and self-adjoint, it possesses a unique positive self-adjoint square root $A_1^{1/2}(q)$ (see [25, p. 281] or [45, p. 197]). Moreover, any positive fractional $\delta$-power $A_1^\delta(q)$ of $A_1(q)$ is well defined, positive, and self-adjoint. It is easy to see that $\mathrm{dom}(B_1^2(q)) = \mathrm{dom}(A_1(q))$ and $B_1^2(q)u = (\beta^2\rho/\gamma)A_1(q)u$ for all $u \in \mathrm{dom}(A_1(q))$. Hence $B_1(q) = (\beta\sqrt{\rho}/\sqrt{\gamma})\, A_1^{1/2}(q)$ and this completes the proof of Theorem 3.3. ∎

We now define the Hilbert space $E_q$ by

$$E_q \doteq \mathrm{dom}(A_1^{1/2}(q)) \times L_{2,\rho}(\Omega) = \mathrm{dom}(B_1(q)) \times L_{2,\rho}(\Omega)$$

with inner product

$$\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle_{E_q} \doteq \langle A_1^{1/2}(q)f_1, A_1^{1/2}(q)f_2\rangle_{L_{2,\rho}} + \langle g_1, g_2\rangle_{L_{2,\rho}},$$

and the operator $C_1(q) : E_q \to E_q$ by $\mathrm{dom}(C_1(q)) \doteq \mathrm{dom}(A_1(q)) \times \mathrm{dom}(A_1^{1/2}(q)) = \mathrm{dom}(A_1(q)) \times \mathrm{dom}(B_1(q))$ and

$$C_1(q) \doteq \begin{pmatrix} 0 & I \\ -A_1(q) & -B_1(q) \end{pmatrix}. \tag{3.17}$$

Note that $\mathrm{dom}(C_1(q))$ is dense in $E_q$. The operator $C_1(q)$ corresponds to the elastic model $\ddot{x} + B_1(q)\dot{x} + A_1(q)x = 0$ written as a first order system. By Theorem 3.3, the elastic operator $A_1(q)$ is positive and self-adjoint on $L_{2,\rho}(\Omega)$. The same is true for the dissipation operator $B_1(q)$.

THEOREM 3.4. *Let $C_1(q) : \mathrm{dom}(C_1(q)) \subset E_q \to E_q$ be as defined above. Then $C_1(q)$ is the infinitesimal generator of a strongly continuous semigroup of contractions $e^{C_1(q)t}$ on $E_q$.*

*Proof.* Let $\eta = \binom{u}{v} \in \mathrm{dom}(C_1(q))$. Then $u \in \mathrm{dom}(A_1(q))$, $v \in \mathrm{dom}(B_1(q))$, and

$$\begin{aligned}
\langle C_1(q)\eta, \eta\rangle_{E_q} &= \left\langle \begin{pmatrix} 0 & I \\ -A_1(q) & -B_1(q) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{E_q} \\
&= \left\langle \begin{pmatrix} v \\ -A_1(q)u - B_1(q)v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{E_q} \\
&= \langle A_1^{1/2}(q)v, A_1^{1/2}(q)u\rangle_{L_{2,\rho}} + \langle -A_1(q)u - B_1(q)v, v\rangle_{L_{2,\rho}}.
\end{aligned}$$

Also, since $A_1^{1/2}(q)$ is self-adjoint, $u \in \mathrm{dom}(A_1(q))$ and $B_1(q)$ is positive, it follows that

$$\begin{aligned}
\langle C_1(q)\eta, \eta\rangle_{E_q} &= \langle v, A_1(q)u\rangle_{L_{2,\rho}} - \langle A_1(q)u, v\rangle_{L_{2,\rho}} - \langle B_1(q)v, v\rangle_{L_{2,\rho}} \\
&= -\langle B_1(q)v, v\rangle_{L_{2,\rho}} \le 0.
\end{aligned}$$

Hence $C_1(q)$ is dissipative. One can easily verify that the adjoint $C_1^*(q)$ of $C_1(q)$ is given by $\mathrm{dom}(C_1^*(q)) = \mathrm{dom}(C_1(q)) = \mathrm{dom}(A_1(q)) \times \mathrm{dom}(B_1(q))$ and

$$C_1^*(q) = \begin{pmatrix} 0 & -I \\ A_1(q) & -B_1(q) \end{pmatrix}.$$

Moreover, if $\eta = \binom{u}{v} \in \mathrm{dom}(C_1^*(q))$, we have

$$\langle C_1^*(q)\eta, \eta\rangle_{E_q} = \left\langle \begin{pmatrix} 0 & -I \\ A_1(q) & -B_1(q) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{E_q}$$

$$= \left\langle \begin{pmatrix} -v \\ A_1(q)u - B_1(q)v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{E_q}$$
$$= \langle A_1^{1/2}(q)v, A_1^{1/2}(q)u\rangle_{L_{2,\rho}} + \langle A_1(q)u - B_1(q)v, v\rangle_{L_{2,\rho}}.$$

Again, since $A_1^{1/2}(q)$ is self-adjoint, $u \in \operatorname{dom}(A_1(q))$ and $B_1(q)$ is positive, it follows that

$$\langle C_1^*(q)\eta, \eta\rangle_{E_q} = -\langle v, A_1(q)u\rangle_{L_{2,\rho}} + \langle A_1(q)u, v\rangle_{L_{2,\rho}} - \langle B_1(q)v, v\rangle_{L_{2,\rho}}$$
$$= -\langle B_1(q)v, v\rangle_{L_{2,\rho}} \leq 0.$$

Hence $C_1^*(q)$ is also dissipative. It now follows from the Lummer–Phillips theorem (see [35, Corollary 4.4, p. 15]) that $C_1(q)$ is the infinitesimal generator of a strongly continuous semigroup of contractions $e^{C_1(q)t}$ on $E_q$.

*Comment.* Operators of the form $\mathcal{A}_B = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix}$, where both $A$ and $B$ are positive and self-adjoint, appear often in the theory of elastic systems. In fact, they correspond to the elastic model $\ddot{x} + B\dot{x} + Ax = 0$, written as a first order system. In 1982, Russell and Chen [10] first conjectured the analyticity of the $C_0$- semigroup of contractions $e^{\mathcal{A}_B t}$ generated by $\mathcal{A}_B$, when $B$ is "related" to the $\delta$-power of $A$, $0 < \delta \leq 1$. This conjecture was later proved by Triggiani and Chen [12]. It turns out that if $\rho_1 A^\delta \leq B \leq \rho_2 A^\delta$ for some constants $\rho_1, \rho_2$, $0 < \rho_1 < \rho_2 < \infty$, then $e^{\mathcal{A}_B t}$ is analytic if $\frac{1}{2} \leq \delta \leq 1$ and it is not analytic if $0 < \delta < \frac{1}{2}$.

For fixed $q \in \mathcal{Q}$ the eigenvalues of the operator $A_1(q)$ are easily found to be

$$\mu_n = \mu_n(q) = \frac{\gamma n^4 \pi^4}{\rho}, \qquad n = 1, 2, \ldots, \tag{3.18}$$

with corresponding normalized eigenfunctions in $L_{2,\rho}(\Omega)$ given by

$$h_n(x) = \sqrt{\frac{2}{\rho}} \sin(\pi n x). \tag{3.19}$$

THEOREM 3.5. *Let $q \in \mathcal{Q}$, $C_1(q)$ as in (3.17) and $\{\mu_n\}_{n=1}^\infty$ the eigenvalues of $A_1(q)$ given by (3.18). Then,*

(a) *the strongly continuous semigroup of contractions $e^{C_1(q)t}$ generated by $C_1(q)$ on $E_q$ on $E_q$ is also analytic.*

(b) *The spectrum $\sigma(C_1(q))$ of $C_1(q)$ consists only of eigenvalues $\{\lambda_n^{+,-}\}_{n=1}^\infty$, which are the solutions of the equation $\lambda^2 + 2r(q)\mu_n^{1/2}\lambda + \mu_n = 0$, where $r(q) = \beta\sqrt{\rho}/2\sqrt{\gamma}$ and are given by $\lambda_n^{+,-} = \sqrt{\mu_n}\,(-r(q) \pm$*

$\sqrt{r^2(q) - 1})$. *The eigenvalues are real if and only if* $r(q) \geq 1$, *i.e.*, $\beta^2\rho \geq 4\gamma$. *If* $r(q) < 1$, *the eigenvalues lay symmetrically with respect to the real axis on the two rays* $\{xe^{\pm i\alpha(q)},\ 0 \leq x < \infty\}$ *where* $e^{\pm i\alpha(q)} = -r(q) \pm i\sqrt{1 - r^2(q)}$ (*note that* $\alpha(q) > \pi/2$). *In any case,* $\operatorname{Re}\lambda_n^{+,-} < 0$ *for all* $n$,

$$\left|\frac{\operatorname{Im}\lambda_n^{+,-}}{\operatorname{Re}\lambda_n^{+,-}}\right| \leq M(q) \doteq \begin{cases} 0 & \text{if } r(q) \geq 1, \\ \dfrac{\sqrt{1 - r^2(q)}}{r(q)} & \text{if } r(q) < 1, \end{cases}$$

*and the spectrum* $\sigma(C_1(q))$ *of* $C_1(q)$ *is contained in a triangular sector of the form* $\Sigma \doteq \{\lambda \in \mathbb{C} \mid |\arg(\lambda)| > \pi/2 + \theta_0(q)\}$, *where* $\theta_0(q)$ *is any number satisfying* $0 < \theta_0(q) < \pi/2$ *if* $r(q) \geq 1$, *and* $0 < \theta_0(q) < \alpha(q) - \pi/2$ *if* $r(q) < 1$. *The corresponding family of normalized eigenvectors* $\{\phi_n^{+,-}\}_{n=1}^{\infty}$ *on* $E_q$ *is given by*

$$\phi_n^+ = \begin{pmatrix} e_n \\ \lambda_n^+ e_n \end{pmatrix}, \qquad \phi_n^- = k_n \begin{pmatrix} e_n \\ \lambda_n^- e_n \end{pmatrix},$$

*where* $e_n(x) = (2/\rho(\mu_n + |\lambda_n^+|^2)^{1/2} \sin(\pi n x)$, *and* $k_n^2 = (\mu_n + |\lambda_n^+|^2)/(\mu_n + |\lambda_n^-|^2)$. *Note that* $\mu_n$, $\lambda_n^{+,-}$, $\theta_0$, $e_n$, $\phi_n^{+,-}$, *all depend on* $q \in \mathfrak{Q}$.

(c) *The eigenvectors* $\{\phi_n^{+,-}\}_{n=1}^{\infty}$ *satisfy*

(i) $\{\phi_n^+\}_{n=1}^{\infty}$ *is an orthonormal family on* $E_q$;

(ii) $\{\phi_n^-\}_{n=1}^{\infty}$ *is an orthonormal family on* $E_q$, *and*

$$\text{(iii)}\quad \langle \phi_m^+, \phi_n^- \rangle_{E_q} = \begin{cases} 0 & \text{if } n \neq m, \\ k_n(\mu_n + \lambda_n^+ \bar{\lambda}_n^-)\|e_n\|^2_{L_{2,\rho}} & \text{if } m = n \text{ and } \lambda_n^+ \neq \lambda_n^-, \\ 1 & \text{if } m = n \text{ and } \lambda_n^+ = \lambda_n^-. \end{cases}$$

(d) *The eigenvalues of* $C_1^*(q)$ *are* $\{\bar{\lambda}_n^{+,-}\}_{n=1}^{\infty}$, *the conjugates of the eigenvalues of* $C_1(q)$, *with corresponding normalized eigenvectors on* $E_q$ *given by*

$$\phi_m^{*+} = \begin{pmatrix} e_m \\ -\bar{\lambda}_m^+ e_m \end{pmatrix}, \qquad \phi_m^{*-} = k_m \begin{pmatrix} e_m \\ -\bar{\lambda}_m^- e_m \end{pmatrix}.$$

(e) *Assume, in addition that* $\beta^2\rho \neq 4\gamma$ (*or equivalently*, $r(q) \neq 1$) *and let* $\psi_m^{*+} \doteq (1/v_m^+)\ \phi_m^{*+}$, $\psi_m^{*-} \doteq (1/v_m^- k_m)\ \phi_m^{*-}$, *where* $v_m^- \doteq (\mu_m - (\bar{\lambda}_m^-)^2)/(\mu_m + |\lambda_m^-|^2)$, $v_m^+ \doteq (\mu_m - (\bar{\lambda}_m^+)^2)/(\mu_m + |\lambda_m^+|^2)$. *Then the non-normalized eigenvectors* $\{\psi_m^{*+,*-}\}_{m=1}^{\infty}$ *form a biorthogonal system with respect to the eigenvectors* $\{\phi_m^{+,-}\}_{m=1}^{\infty}$ *of* $C_1(q)$, *in the sense that*

$$\langle \psi_m^{*+}, \phi_n^+ \rangle_{E_q} = \langle \psi_m^{*-}, \phi_n^- \rangle_{E_q} = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

*and*

$$\langle \psi_m^{*-}, \phi_n^+ \rangle_{E_q} = \langle \psi_m^{*+}, \phi_n^- \rangle_{E_q} = 0 \qquad \text{for all } m, n.$$

(f) *The operator $C_1(q)$ and the semigroup $e^{C_1(q)t}$ generated by $C_1(q)$ on $E_q$ have the following spectral representations*:

$$C_1(q)\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{n=1}^{\infty} \lambda_n^+ \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \psi_n^{*+} \right\rangle_{E_q} \phi_n^+ + \sum_{n=1}^{\infty} \lambda_n^- \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \psi_n^{*-} \right\rangle_{E_q} \phi_n^-,$$

$$e^{C_1(q)t}\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{n=1}^{\infty} e^{\lambda_n^+ t} \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \psi_n^{*+} \right\rangle_{E_q} \phi_n^+ + \sum_{n=1}^{\infty} e^{\lambda_n^- t} \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \psi_n^{*-} \right\rangle_{E_q} \phi_n^-.$$

(g) *Finally, the semigroup $e^{C_1(q)t}$ satisfies the stability condition $\|e^{C_1(q)t}\|_{\mathcal{L}(E_q)} \leq e^{-\varepsilon(q)t}$ where $\varepsilon(q) = -\sup_{\lambda \in \sigma(C_1(q))} \operatorname{Re} \lambda = -\operatorname{Re} \lambda_1^+$ is explicitly given by*

$$\varepsilon(q) = \begin{cases} \sqrt{\mu_1}\, r(q) = \beta\pi^2/2, & \text{if } \beta^2\rho \leq 4\gamma, \\ \sqrt{\mu_1}\,(r(q) - \sqrt{r^2(q) - 1}) = (\pi^2/2\sqrt{\rho})(\beta\sqrt{\rho} - \sqrt{\beta^2\rho - 4\gamma}), \\ \qquad \text{if } \beta^2\rho > 4\gamma. \end{cases}$$

*Proof.* The proof of this theorem follows immediately from Theorems 1.1 and 1.2 and Lemmas A1 and A2 in [12], by noting that the operator $C_1(q)$ corresponds to the elastic system $\ddot{x} + 2r(q)A_1^{1/2}(q)\dot{x} + A_1(q)x = 0$ written as a first order system. ∎

Let $L_{2,(C_v/k)}(\Omega)$ denote the Hilbert space $L_2(\Omega)$ with inner product $\langle w, \hat{w} \rangle_{L_{2,C_v/k}} \doteq (C_v/k) \int_\Omega w\hat{w}$ and define the operator $C_2(q)$ on $L_{2,C_v/k}(\Omega)$ by

$$\operatorname{dom}(C_2(q)) \doteq \{w \in H^2(\Omega) \mid w'(0) = 0,\ kw'(1) + k_1 w(1) = 0\},$$

$$C_2(q) \doteq \frac{k}{C_v} \frac{\partial^2}{\partial x^2}. \tag{3.20}$$

THEOREM 3.6. *Let $C_2(q) : \operatorname{dom}(C_2(q)) \subset L_{2,C_v/k}(\Omega) \to L_{2,C_v/k}(\Omega)$ be as defined above. Then,*

(i) *$C_2(q)$ is dissipative and self-adjoint*;

(ii) *The spectrum* $\sigma(C_2(q))$ *consists only of eigenvalues* $\{\alpha_n\}_{n=1}^{\infty}$ *given by* $\alpha_n = -k\tau_n^2/C_v$, $n = 1, 2, \ldots$, *where* $\{\tau_n\}_{n=1}^{\infty}$ *are all the positive solutions of the equation* $\tan \tau = k_1/k\tau$. *The corresponding normalized eigenfunctions* $\{\chi_n\}_{n=1}^{\infty}$ *in* $L_{2,C_v/k}(\Omega)$ *are given by* $\chi_n(x) = (k\tau_n/C_v \int_0^{\tau_n}\cos^2(\xi)\, d\xi)^{1/2} \cos(\tau_n x)$;

(iii) *The operator* $C_2(q)$ *generates a strongly continuous semigroup of contractions* $e^{C_2(q)t}$ *on* $L_{2,C_v/k}(\Omega)$ *which is also analytic and satisfies the stability property* $\|e^{C_2(q)t}\|_{\mathcal{L}(L_{2,C_v/k}(\Omega))} \leq e^{-(k\tau_1^2/C_v)t}$, *for all* $t \geq 0$;

(iv) *The semigroup* $e^{C_2(q)t}$ *has the spectral representation* $e^{C_2(q)t}w = \sum_{n=1}^{\infty} e^{\alpha_n t} \langle w, \chi_n\rangle_{L_{2,C_v/k}} \chi_n$.

*Proof.* (i) Using integration by parts and the boundary conditions for $C_2(q)$, it follows that $\langle C_2(q)w, w\rangle_{L_{2,C_v/k}} = -(k_1/k)\,[w(1)]^2 - \|w'\|_{L_2}^2$ for any $w \in \mathrm{dom}\,(C_2(q))$. Thus $C_2(q)$ is dissipative. Similarly, one can easily show that $C_2(q)$ is symmetric. Moreover, an application of the Fundamental Lemma of the Calculus of Variations yields that $C_2(q)$ is self-adjoint. Since the steps are exactly the same as those in the proof of Theorem 3.2, we do not give details here.

(ii) The function $f$ is an eigenvector of $C_2(q)$ corresponding to the eigenvalue $\xi$ if and only if $f \neq 0$ satisfies $f''(x) = (C_v\xi/k)f(x)$, $f'(0) = 0$, and $kf'(1) + k_1 f(1) = 0$. This implies that $\xi < 0$ satisfies $k/k_1 \sqrt{-(C_v\xi/k)} \tan (\sqrt{-(C_v\xi/k)}) = 1$ and $f(x) = b \cos(\sqrt{-(C_v\xi/k)}\, x)$, $b \in \mathbb{R}$. Letting $\tau = \sqrt{-C_v\xi/k}$ we get that the eigenvalues $\{\alpha_n\}_{n=1}^{\infty}$ of $C_2(q)$ are given by $\alpha_n = -k\tau_n^2/C_v$ where $\{\tau_n\}_{n=1}^{\infty}$ are all the positive solutions of the equation $(k/k_1)\,\tau \tan \tau = 1$. Moreover, normalization of $\chi_n(x) = b_n \cos(\tau_n x)$ gives $b_n = (k\tau_n/C_v \int_0^{\tau_n} \cos^2(\xi)\, d\xi)^{1/2}$.

(iii, iv) Since $C_2(q)$ is self-adjoint with a pure point spectrum it follows that (see [45]) $L_{2,C_v/k}(\Omega) = \overline{\mathrm{span}}\,\{\chi_n\}_{n=1}^{\infty}$, $w = \sum_{n=1}^{\infty} \langle w, \chi_n\rangle_{L_{2,C_v/k}} \chi_n$ for any $w \in L_{2,C_v/k}(\Omega)$, $C_2(q)w = \sum_{n=1}^{\infty} \alpha_n \langle w, \chi_n\rangle_{L_{2,C_v/k}} \chi_n$ for any $w \in \mathrm{dom}(C_2(q))$, and $e^{C_2(q)t}w = \sum_{n=1}^{\infty} e^{\alpha_n t} \langle w, \chi_n\rangle_{L_{2,C_v/k}} \chi_n$ for any $w \in L_{2,C_v/k}(\Omega)$. From this representation for $e^{C_2(q)t}$ it is easy to see that $\|e^{C_2(q)t}\|_{\mathcal{L}(L_{2,C_v/k}(\Omega))} \leq e^{\alpha_1 t} = e^{-(k\tau_1^2/C_v)t}$ for all $t \geq 0$. ∎

We note now that $Z_q$ is isometrically isomorphic to $E_q \times L_{2,C_v/k}(\Omega)$. In fact, $\eta = \binom{u}{v} \in E_q$ and $w \in L_{2,C_v/k}(\Omega)$ if and only if

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q.$$

Also note that for any $\eta = \binom{u}{v}$, $\hat{\eta} = \binom{\hat{u}}{\hat{v}} \in E_q$, $w$, $\hat{w} \in L_{2,C_v/k}(\Omega)$, we have

$$
\begin{aligned}
\left\langle \begin{pmatrix} \eta \\ w \end{pmatrix}, \begin{pmatrix} \hat{\eta} \\ \hat{w} \end{pmatrix} \right\rangle_{E_q \times L_{2,C_v/k}} &= \langle \eta, \hat{\eta} \rangle_{E_q} + \langle w, \hat{w} \rangle_{L_{2,C_v/k}} \\
&= \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \right\rangle_{E_q} + \langle w, \hat{w} \rangle_{L_{2,C_v/k}} \\
&= \langle A_1^{1/2} u, A_1^{1/2} \hat{u} \rangle_{L_{2,\rho}} + \langle v, \hat{v} \rangle_{L_{2,\rho}} + \langle w, \hat{w} \rangle_{L_{2,C_v/k}} \\
&= \rho \int_\Omega \sqrt{\frac{\gamma}{\rho}}\, u'' \sqrt{\frac{\gamma}{\rho}}\, \hat{u}'' + \rho \int_\Omega v\hat{v} + \frac{C_v}{k} \int_\Omega w\hat{w} \\
&= \gamma \int_\Omega u''\hat{u}'' + \rho \int_\Omega v\hat{v} + \frac{C_v}{k} \int_\Omega w\hat{w} \\
&= \left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} \right\rangle_{Z_q}.
\end{aligned}
$$

Thus, the Hilbert space $E_q \times L_{2,C_v/k}(\Omega)$ endowed with the usual inner product is isometrically isomorphic to $Z_q$. From now on we will freely make use of this identification.

The following identity relating the operators $A(q)$, $C_1(q)$, and $C_2(q)$ can be immediately verified,

$$
A(q) = \begin{pmatrix} C_1(q) & 0 \\ 0 & C_2(q) \end{pmatrix},
$$

where the equality should be interpreted in the sense of the isomorphism previously defined. This identity allows us to use Theorems 3.5 and 3.6 from which we obtain the following.

THEOREM 3.7. *Let* $A(q)$: $\mathrm{dom}(A(q)) \subset Z_q \to Z_q$ *be as in* (3.6), (3.7). *Then*

(i) $0 \in \rho(A(q))$, *the resolvent set of* $A(q)$;

(ii) *the spectrum* $\sigma(A(q))$ *of* $A(q)$ *consists only of eigenvalues,*

$$
\sigma(A(q)) = \sigma_p(A(q)) = \sigma_p(C_1(q)) \cup \sigma_p(C_2(q)) = \{\lambda_n^{+,-}, \alpha_n\}_{n=1}^{\infty}
$$

*where* $\lambda_n^{+,-} = \sqrt{\mu_n}\,(-r(q) \pm \sqrt{r^2(q) - 1})$, $\alpha_n = -k\tau_n^2/C_v$, *with* $\mu_n = \gamma n^4 \pi^4/\rho$, $r(q) = \beta\sqrt{\rho}/2\sqrt{\gamma}$, *and* $\{\tau_n\}_{n=1}^{\infty}$ *are all the positive solutions of the equation* $\tan\tau = k_1/k\tau$. *The corresponding set of normalized eigenvectors in* $Z_q$ *is given by*

$$\left\{\begin{pmatrix} e_n \\ \lambda_n^+ e_n \\ 0 \end{pmatrix}, \begin{pmatrix} k_n e_n \\ k_n \lambda_n^- e_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \chi_n \end{pmatrix}\right\}_{n=1}^{\infty},$$

*where*

$$e_n(x) = \left(\frac{2}{\rho(\mu_n + |\lambda_n^+|^2)}\right)^{1/2} \sin(\pi n x),$$

$$\chi_n(x) = \left(\frac{k\tau_n}{C_v \int_0^{\tau_n} \cos^2(\xi)\, d\xi}\right)^{1/2} \cos(\tau_n x)$$

*and* $k_n^2 = (\mu_n + |\lambda_n^+|^2)/(\mu_n + |\lambda_n^-|^2)$.

(iii) *The operator* $A(q)$ *generates an exponentially stable analytic semigroup* $T(t;\, q)$ *which satisfies* $\|T(t;\, q)\|_{\mathscr{L}(Z_q)} \le e^{-\omega(q)t}$, *for* $t \ge 0$, *where* $\omega(q)$ *is given by*

$$\omega(q) = \begin{cases} \min\left(\dfrac{k\tau_1^2}{C_v}, \dfrac{\beta\pi^2}{2}\right), & \text{if } \beta^2\rho \le 4\gamma. \\[2ex] \min\left(\dfrac{k\tau_1^2}{C_v}, \dfrac{\beta\pi^2}{2} - \dfrac{\pi^2}{2\sqrt{\rho}}\sqrt{\beta^2\rho - 4\gamma}\right), & \text{if } \beta^2\rho > 4\gamma. \end{cases}$$

*Moreover, the semigroup* $T(t,\, q)$ *has the representation*

$$\begin{aligned} T(t;\, q)z &= \sum_{n=1}^{\infty} e^{\lambda_n^+ t} \left\langle z, \frac{1}{v_n^+}\begin{pmatrix} e_n \\ -\overline{\lambda_n^+}\, e_n \\ 0 \end{pmatrix}\right\rangle_q \begin{pmatrix} e_n \\ \lambda_n^+ e_n \\ 0 \end{pmatrix} \\ &+ \sum_{n=1}^{\infty} e^{\lambda_n^- t} \left\langle z, \frac{1}{v_n^-}\begin{pmatrix} e_n \\ -\overline{\lambda_n^-}\, e_n \\ 0 \end{pmatrix}\right\rangle_q \begin{pmatrix} k_n e_n \\ k_n \lambda_n^- e_n \\ 0 \end{pmatrix} \\ &+ \sum_{n=1}^{\infty} e^{\alpha_n t} \left\langle z, \begin{pmatrix} 0 \\ 0 \\ \chi_n \end{pmatrix}\right\rangle_q \begin{pmatrix} 0 \\ 0 \\ \chi_n \end{pmatrix}, \end{aligned}$$

*where* $v_n^{+,-} = (\mu_n - (\overline{\lambda_n^{+,-}})^2)/(\mu_n + |\overline{\lambda_n^{+,-}}|^2)$.

*Proof.* It can be immediately verified that $A(q)$ is invertible and

$$A^{-1}(q) = \begin{pmatrix} C_1^{-1}(q) & 0 \\ 0 & C_2^{-1}(q) \end{pmatrix},$$

where

$$C_1^{-1}(q) = \begin{pmatrix} -B_1(q)A_1^{-1}(q) & -A_1^{-1}(q) \\ I & 0 \end{pmatrix} = \begin{pmatrix} -\dfrac{\beta\sqrt{\rho}}{\sqrt{\gamma}} A_1^{-1/2}(q) & -A_1^{-1}(q) \\ I & 0 \end{pmatrix}.$$

Parts (ii) and (iii) follow immediately from Theorems 3.5 and 3.6 and the fact that $C_1(q)$ and $C_2(q)$ "decouple" the operator $A(q)$. ∎

Since $A(q)$ is closed, dom($A(q)$) endowed with the norm of the graph of $A(q)$, $\||z\||_{A(q)} \doteq \|z\|_q + \|A(q)z\|_q$ is a Hilbert space which we denote by $Z_{A(q)}$, i.e., $Z_{A(q)} \doteq (\mathrm{dom}(A(q)); \||\cdot\||_{A(q)})$. Moreover, since $A(q)$ is invertible, $\||\cdot\||_{A(q)}$ is equivalent to $\||\cdot\||_G$ defined by $\||z\||_G = \|A(q)z\|_q$ on dom($A(q)$). Now, for

$$z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathrm{dom}(A(q))$$

we define $\|z\|_{H^2}^2 \doteq \|u\|_{H^2}^2 + \|v\|_{H^2}^2 + \|w\|_{H^2}^2$, while $\|z\|_{H^1}^2$ is defined in the analogous way. The following lemma relates the $H^2$-norm and the $\||\cdot\||_{A(q)}$-norm on dom($A(q)$).

LEMMA 3.8. *Let $\mathcal{S}$ be a bounded subset of $Z_{A(q)}$. Then $\mathcal{S}$ is also bounded in the $H^2$-norm, i.e., there exists a constant $C$ depending on $q$ and $\mathcal{S}$ such that $\|z\|_{H^2}^2 \le C$ for all $z \in \mathcal{S}$. Moreover, the constant $C$ can be chosen independent of $q$ on compact subsets of $\mathcal{Q}$.*

*Proof.* Since $\mathcal{S}$ is bounded in $Z_{A(q)}$, there exists $M > 0$ such that $\||z\||_{A(q)}^2 \le M$ for all $z \in \mathcal{S}$. Let

$$z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{S}.$$

Then, $\|u\|_{L_2} \le \|u'\|_{L_2} \le \|u''\|_{L_2}$, since $u \in H_0^1(0, 1) \cap H^2(0, 1)$. The first inequality, known as Poincaré's first inequality (see [47, p. 116]), follows immediately from Schwartz's inequality and the fact that $u(0) = 0$. For the second one, note that $\|u'\|_{L_2}^2 = \int_0^1 (u')^2 = -\int_0^1 uu'' \le \|u\|_{L_2}\|u''\|_{L_2} \le \|u'\|_{L_2}\|u''\|_{L_2}$. Hence, $\|u'\|_{L_2} \le \|u''\|_{L_2}$.

Now $\|u''\|_{L_2}^2 \le \|z\|_q^2/\gamma \le \||z\||_{A(q)}^2/\gamma \le M/\gamma$. Similarly we have $\|v\|_{L_2} \le \|v'\|_{L_2} \le \|v''\|_{L_2}$, (since $v \in H_0^1(0, 1) \cap H^2(0, 1)$) and $\|v''\|_{L_2}^2 \le \|A(q)z\|_q^2/\gamma \le \||z\||_{A(q)}^2/\gamma \le M/\gamma$. Finally, we have

$$\|w\|_{L_2}^2 \le \frac{\|z\|_q^2}{C_v/k} \le \frac{k}{C_v} |||z|||_{A(q)}^2 \le \frac{Mk}{C_v},$$

$$\|w''\|_{L_2}^2 = \frac{C_v}{k}\left(\frac{C_v}{k}\left\|\frac{k}{C_v} w''\right\|_{L_2}^2\right) \le \frac{C_v}{k}\|A(q)z\|_q^2 \le \frac{C_v}{k}|||z|||_{A(q)}^2 \le \frac{C_v M}{k},$$

$$\|w'\|_{L_2}^2 = w'w|_{\partial\Omega} - \int_\Omega ww'' \le w'(1)w(1) + \|w\|_{L_2}\|w''\|_{L_2}$$

$$= -\frac{k_1}{k}[w(1)]^2 + \|w\|_{L_2}\|w''\|_{L_2} \le \|w\|_{L_2}\|w''\|_{L_2}$$

$$\le \sqrt{\frac{kM}{C_v}}\sqrt{\frac{C_v M}{k}} = M.$$

Hence, $\|z\|_{H^2}^2 \le C$, where $C = M(6/\gamma + C_v/k + k/C_v + 1)$ can be chosen independent of $q$ on any subset $\mathcal{Q}_S$ of $\mathcal{Q}$ of the form

$$\mathcal{Q}_S = \{q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \in \mathcal{Q} \mid 0 < a \le C_v \le b < \infty, 0 < c \le \gamma\},$$

and therefore on any compact subset of $\mathcal{Q}$. ∎

*Remark.* By Theorem 3.7, $A(q)$ is the infinitesimal generator of an analytic semigroup and $0 \in \rho(-A(q))$. Hence, the fractional $\delta$-powers of $-A(q)$, $[-A(q)]^\delta$ are well defined, closed, linear, invertible operators for all $0 < \delta \le 1$ (see [35, Section 2.6]). Therefore, $\text{dom}([-A(q)]^\delta)$ endowed with the graph norm $|||z|||_{A^\delta(q)} \doteq \|z\|_q + \|[-A(q)]^\delta z\|_q$ is a Hilbert space, which we denote by $Z_{A^\delta(q)}$.

One way of showing that the initial value problem (3.5) is well-posed involves proving that the nonlinear term $F(q, t, z)$ is sufficiently regular with respect to the operator $A(q)$. More precisely, one of the requirements is that $F(q, t, z)$ be locally uniformly Hölder continuous in $z$ with respect to the norm of the graph of some fractional $\delta$-power of $-A(q)$.

DEFINITION OF CONDITION (F). Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$ on a Banach space $X$ and assume $0 \in \rho(-A)$. For $0 < \delta \le 1$, let $X_\delta$ denote the Banach space $(\text{dom}(A^\delta); \|\cdot\|_\delta)$, where $\|x\|_\delta \doteq \|A^\delta x\|$. Let $U$ be an open subset of $\mathbb{R}^+ \times X_\delta$. We say that the function $f: U \to X$ satisfies the condition $(F)$ on $U$ if for every $(t, x) \in U$ there exists a neighborhood $V \subset U$ and constants $L \ge 0$, $0 < \nu \le 1$, such that $\|f(t_1, x_1) - f(t_2, x_2)\| \le L(|t_1 - t_2|^\nu + \|x_1 - x_2\|_\delta)$ for all $(t_1, x_1), (t_2, x_2) \in V$, i.e., if $f$ is locally Hölder continuous in $t$ and locally Lipschitzian in $x$, on $U$.

THEOREM 3.9. *Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$ satisfying $\|T(t)\| \leq M$ and assume further that $0 \in \rho(-A)$. If $f$ satisfies the condition $(F)$ for some $0 < \delta \leq 1$, then for every initial data $(t_0, x_0) \in U$ the initial value problem*

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = f(t, u(t)), & t > 0, \\ u(t_0) = x_0 \end{cases}$$

*has a unique local (strong) solution $u \in C([t_0, t_1); X) \cap C^1((t_0, t_1); X)$ where $t_1 = t_1(t_0, x_0) > t_0$.*

*Proof.* The proof of this theorem for the case $0 < \delta < 1$ can be found in [35, p. 196]. In the case $\delta = 1$, it is enough that $-A$ be the infinitesimal generator of a $C_0$-semigroup, i.e., one can weaken the analyticity assumption. The proof for this particular case can be found in [35, p. 190]. ∎

The following theorem states that the nonlinear part of system (3.5) satisfies the condition $(F)$ with $\delta = 1$.

THEOREM 3.10. *Let $q \in \mathcal{Q}$, $U$ be a bounded open neighborhood of the origin in $\mathbb{R}_0^+ \times Z_{A(q)}$, and define $h_q : U \to Z_q$ by $h_q(t, z) = F(q, t, z)$. Then $h_q$ satisfies the condition $(F)$ on $U$ with constants $\delta = 1$ and $\nu = 1$, uniformly in $U$, i.e., there exists a constant $L > 0$ such that*

$$\|F(q, t_1, z_1) - F(q, t_2, z_2)\|_q \leq L(|t_1 - t_2| + |||z_1 - z_2|||_{A(q)}) \tag{3.21}$$

*for all $(t_1, z_1), (t_2, z_2) \in U$. The constant $L$ in general depends on $q$ but it can be chosen independent of $q$ on compact subsets of $\mathcal{Q}$.*

*Proof.* Since $U$ is bounded, there exist constants $M_1, M_2 > 0$ such that $|t| \leq M_1$ and $|||z|||_{A(q)} \leq M_2$ for all $(t, z) \in U$. Also, by Lemma 3.8, there exists a constant $C_1$ depending on $q$ such that $\|z\|_{H^2} \leq C_1$ for all $(t, z) \in U$. More precisely, $C_1 = M_2 \sqrt{6/\gamma + C_v/k + k/C_v + 1}$. Let $(t_i, z_i) \in U$,

$$z_i = \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}, \quad i = 1, 2.$$

Then

$$
\begin{aligned}
\|h_q(t_1, z_1) - h_q(t_2, z_2)\|_q &= \|F(q, t_1, z_1) - F(q, t_2, z_2)\|_q \\
&= \left\| \begin{pmatrix} 0 \\ f_2(q, t_1, z_1) - f_2(q, t_2, z_2) \\ f_3(q, t_1, z_1) - f_3(q, t_2, z_2) \end{pmatrix} \right\|_q \\
&= \left\{ \rho \left\| f_2(q, t_1, z_1) - f_2(q, t_2, z_2) \right\|_{L_2}^2 \right. \\
&\qquad \left. + \frac{C_v}{k} \left\| f_3(q, t_1, z_1) - f_3(q, t_2, z_2) \right\|_{L_2}^2 \right\}^{1/2} \\
&\leq \sqrt{\rho} \left\| f_2(q, t_1, z_1) - f_2(q, t_2, z_2) \right\|_{L_2} \\
&\qquad + \sqrt{\frac{C_v}{k}} \left\| f_3(q, t_1, z_1) - f_3(q, t_2, z_2) \right\|_{L_2}^2 .
\end{aligned}
$$

Therefore, $h_q(t, z)$ satisfies the condition $(F)$ uniformly on $U$ with $\nu = 1$ if both $f_2(q, t, z)$ and $f_3(q, t, z)$ are uniformly Lipschitz in $t$ on $U$ and uniformly Lipschitz in $z$ on $U$ with respect to the $|||\cdot|||_{A(q)}$-norm. Let us work first with $f_3(q, t, z)$. We have

$$
\begin{aligned}
f_3(q, t_1, z_1) - f_3(q, t_2, z_2) &= C_v^{-1} [g(\cdot, t_1) - g(\cdot, t_2)] \\
&\quad + 2\alpha_2 C_v^{-1} [(w_1 + L(\cdot, t_1)) u_1' v_1' - (w_2 + L(\cdot, t_2)) u_2' v_2'] \\
&\quad + \beta\rho C_v^{-1} (v_1'^2 - v_2'^2) \\
&\quad - \cos(2\pi \cdot) [\theta_\Gamma'(t_1) - \theta_\Gamma'(t_2)] \\
&\quad - 4k\pi^2 C_v^{-1} [L(\cdot, t_1) - L(\cdot, t_2)] \\
&\doteq I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}
\tag{3.22}
$$

where $I_i$ is the $i$th-term in the order given above.

Using hypothesis (H1) we obtain

$$
\|I_1\|_{L_2} \leq C_v^{-1} \|K_g\|_{L_2} |t_1 - t_2|. \tag{3.23a}
$$

Also, $\|I_3\|_{L_2} = \beta\rho C_v^{-1} \|v_1'^2 - v_2'^2\|_{L_2} \leq \beta\rho C_v^{-1} (\|v_1'\|_{L_\infty} + \|v_2'\|_{L_\infty}) \|v_1' - v_2'\|_{L_2}$, and using the Sobolev embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ we have $\|v_1'\|_{L_\infty} \leq C_\infty \|v_1'\|_{H^1} \leq C_\infty \|v_1\|_{H^2} \leq C_\infty \|z_1\|_{H^2} \leq C_\infty C_1$, where $C_\infty$ is a positive constant coming from the embedding. Similarly, $\|v_2'\|_{L_\infty} \leq C_\infty C_1$ and, therefore, it follows that $\|I_3\|_{L_2} \leq 2\beta\rho C_v^{-1} C_\infty C_1 \|v_1' - v_2'\|_{L_2}$. Now, since $v_i \in H_0^1 \cap H^2(0, 1)$, $i = 1, 2$, it follows that

$$\begin{aligned} \|I_3\|_{L_2} &\leq 2\beta\rho C_v^{-1} C_\infty C_1 \|v_1'' - v_2''\|_{L_2} \\ &\leq 2\beta\rho C_v^{-1} C_\infty C_1 \frac{1}{\sqrt{\gamma}} \|A(q)(z_1 - z_2)\|_q \\ &\leq \frac{2\beta\rho C_\infty C_1}{C_v\sqrt{\gamma}} \|\|z_1 - z_2\|\|_{A(q)}. \end{aligned} \tag{3.23b}$$

From hypothesis (H2) we get

$$\|I_4\|_{L_2} \leq |\theta_\Gamma'(t_1) - \theta_\Gamma'(t_2)| \leq K_{M_1}' |t_1 - t_2|, \tag{3.23c}$$

where $K_{M_1}'$ is the (local) Lipschitz constant for $\theta_\Gamma'$ on the interval $[0, M_1]$.
Turning to $I_5$, we obtain the bound

$$\begin{aligned} \|I_5\|_{L_2} &= 4k\pi^2 C_v^{-1} |\theta_\Gamma(t_1) - \theta_\Gamma(t_2)| \left( \int_\Omega \cos^2(2\pi x)\, dx \right)^{1/2} \\ &\leq 4k\pi^2 C_v^{-1} K_{M_1} |t_1 - t_2|, \end{aligned} \tag{3.23d}$$

where $K_{M_1}$ is the (local) Lipschitz constant for $\theta_\Gamma$ corresponding to the interval $[0, M_1]$.

We now have to estimate $\|I_2\|_{L_2}$. We separate $I_2$ into two terms, as follows:

$$\begin{aligned} I_2 &= 2\alpha_2 C_v^{-1} [(w_1 u_1' v_1' - w_2 u_2' v_2') + (L(\cdot, t_1) u_1' v_1' - L(\cdot, t_2) u_2' v_2')] \\ &\doteq 2\alpha_2 C_v^{-1} (I_{21} + I_{22}). \end{aligned} \tag{3.24}$$

Writing $I_{21}$ in the form $I_{21} = w_1 v_1'(u_1' - u_2') + u_2' w_1 (v_1' - v_2') + u_2' v_2' (w_1 - w_2)$, we obtain the estimate

$$\begin{aligned} \|I_{21}\|_{L_2} \leq{}& \|w_1\|_{L_\infty} \|v_1'\|_{L_\infty} \|u_1' - u_2'\|_{L_2} + \|u_2'\|_{L_\infty} \|w_1\|_{L_\infty} \|v_1' - v_2'\|_{L_2} \\ &+ \|u_2'\|_{L_\infty} \|v_2'\|_{L_\infty} \|w_1 - w_2\|_{L_2}. \end{aligned}$$

Again, the Sobolev embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ immediately yields $\|w_1\|_{L_\infty} \leq C_\infty C_1$, $\|v_1'\|_{L_\infty} \leq C_\infty C_1$, $\|v_2'\|_{L_\infty} \leq C_\infty C_1$, and $\|u_2'\|_{L_\infty} \leq C_\infty C_1$. Hence,

$$\|I_{21}\|_{L_2} \leq C_\infty^2 C_1^2 (\|u_1' - u_2'\|_{L_2} + \|v_1' - v_2'\|_{L_2} + \|w_1 - w_2\|_{L_2}).$$

Since $u_i, v_i \in H_0^1 \cap H^2$, $i = 1, 2$, it follows that

$$\begin{aligned}\|I_{21}\|_{L_2} &\le C_\infty^2 C_1^2(\|u_1'' - u_2''\|_{L_2} + \|v_1'' - v_2''\|_{L_2} + \|w_1 - w_2\|_{L_2})\\ &\le C_\infty^2 C_1^2\left(\frac{\|z_1 - z_2\|_q}{\sqrt{\gamma}} + \frac{\|A(q)(z_1 - z_2)\|_q}{\sqrt{\gamma}} + \frac{\|z_1 - z_2\|_q}{\sqrt{C_v/k}}\right)\\ &\le C_\infty^2 C_1^2\left(\frac{1}{\sqrt{\gamma}} + \sqrt{\frac{k}{C_v}}\right)|||z_1 - z_2|||_{A(q)}. \qquad (3.25a)\end{aligned}$$

Similarly, we have

$$\begin{aligned}I_{22} &= L(\cdot, t_1)u_1'v_1' - L(\cdot, t_2)u_2'v_2'\\ &= L(\cdot, t_1)v_1'(u_1' - u_2') + L(\cdot, t_1)u_2'(v_1' - v_2') + u_2'v_2'(L(\cdot, t_1) - L(\cdot, t_2))\\ &= \theta_\Gamma(t_1)\cos(2\pi\cdot)v_1'(u_1' - u_2') + \theta_\Gamma(t_1)\cos(2\pi\cdot)u_2'(v_1' - v_2')\\ &\quad + u_2'v_2'\cos(2\pi\cdot)(\theta_\Gamma(t_1) - \theta_\Gamma(t_2)).\end{aligned}$$

Using hypothesis (H2) and the embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$, it follows that

$$\begin{aligned}\|I_{22}\|_{L_2} &\le \|\theta_\Gamma\|_{L_\infty(0,M_1)} C_\infty C_1\|u_1' - u_2'\|_{L_2} + \|\theta_\Gamma\|_{L_\infty(0,M_1)} C_\infty C_1\|v_1' - v_2'\|_{L_2}\\ &\quad + C_\infty^2 C_1^2 K_{M_1}|t_1 - t_2|\\ &\le \|\theta_\Gamma\|_{L_\infty(0,M_1)} C_\infty C_1(\|u_1'' - u_2''\|_{L_2} + \|v_1'' - v_2''\|_{L_2})\\ &\quad + C_\infty^2 C_1^2 K_{M_1}|t_1 - t_2| \qquad (3.25b)\\ &\le \|\theta_\Gamma\|_{L_\infty(0,M_1)} C_\infty C_1\left(\frac{\|z_1 - z_2\|_q}{\sqrt{\gamma}} + \frac{\|A(q)(z_1 - z_2)\|_q}{\sqrt{\gamma}}\right)\\ &\quad + C_\infty^2 C_1^2 K_{M_1}|t_1 - t_2|\\ &\le \frac{C_\infty C_1\|\theta_\Gamma\|_{L_\infty(0,M_1)}}{\sqrt{\gamma}}|||z_1 - z_2|||_{A(q)} + C_\infty^2 C_1^2 K_{M_1}|t_1 - t_2|.\end{aligned}$$

From (3.24) and (3.25a), (3.25b) we get

$$\|I_2\|_{L_2} \le C_2|t_1 - t_2| + C_3|||z_1 - z_2|||_{A(q)}, \qquad (3.26)$$

where $C_2 = 2\alpha_2 C_\infty^2 C_1^2 K_{M_1}/C_v$, and $C_3 = (2\alpha_2/C_v)\ \{C_\infty^2\ C_1^2(1/\sqrt{\gamma} + \sqrt{k/C_v}) + C_\infty C_1\|\theta_\Gamma\|_{L_\infty(0,M_1)}/\sqrt{\gamma}\}$.

Substituting (3.23a)–(3.23d) and (3.26) into (3.22), we finally obtain the estimate

$$\|f_3(q, t_1, z_1) - f_3(q, t_2, z_2)\|_{L_2} \le C_4|t_1 - t_2| + C_5|||z_1 - z_2|||_{A(q)}, \quad (3.27)$$

where $C_4 = \|K_g\|_{L_2}/C_v + C_2 + K'_{M_1} + 4k\pi^2 K_{M_1}/C_v$ and $C_5 = C_3 + 2\beta\rho C_\infty C_1/C_v\sqrt{\gamma}$.

We turn now to $f_2(q, t, z)$. We have that

$$
\begin{aligned}
f_2(q, t_1, z_1) - f_2(q, t_2, z_2) &= \rho^{-1}[f(\cdot, t_1) - f(\cdot, t_2)] \\
&\quad + \rho^{-1}\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial \varepsilon}\Psi\,(u_1', u_1'', w_1 + L(\cdot, t_1))\right. \\
&\quad \left. - \frac{\partial}{\partial \varepsilon}\Psi\,(u_2', u_2'', w_2 + L(\cdot, t_2))\right\} \\
&= \rho^{-1}[f(\cdot, t_1) - f(\cdot, t_2)] \\
&\quad + \rho^{-1}\frac{\partial}{\partial x}\{2\alpha_2\,[(w_1 + L(\cdot, t_1) - \theta_1)\,u_1' \\
&\quad - (w_2 + L(\cdot, t_2) - \theta_1)u_2'] \\
&\quad -4\alpha_4(u_1'^3 - u_2'^3) + 6\alpha_6\,(u_1'^5 - u_2'^5)\} \qquad (3.28) \\
&= \rho^{-1}\,[f(\cdot, t_1) - f(\cdot, t_2)] + 2\alpha_2\rho^{-1}\,(w_1'u_1' - w_2'u_2') \\
&\quad + 2\alpha_2\rho^{-1}\left(\frac{\partial}{\partial x}L(\cdot, t_1)u_1' - \frac{\partial}{\partial x}L(\cdot, t_2)u_2'\right) \\
&\quad + 2\alpha_2\rho^{-1}\,(w_1u_1'' - w_2u_2'') + 2\alpha_2\rho^{-1}\,[L(\cdot, t_1)u_1'' - L(\cdot, t_2)u_2''] \\
&\quad - 2\alpha_2\rho^{-1}\theta_1\,(u_1'' - u_2'') - 12\alpha_4\rho^{-1}\,(u_1'^2u_1'' - u_2'^2u_2'') \\
&\quad + 30\alpha_6\rho^{-1}\,(u_1'^4u_1'' - u_2'^4u_2'') \\
&\doteq \sum_{i=1}^{8} T_i,
\end{aligned}
$$

where $T_i$, $i = 1, 2, \ldots, 8$, represent the above terms in their respective orders. We shall obtain estimates on each one of the $T_i$'s. Using the hypothesis (H1), it follows that

$$\|T_1\|_{L_2} \leq \rho^{-1}\,\|K_f\|_{L_2}\,|t_1 - t_2|. \tag{3.29a}$$

To estimate $T_2$ we write

$$
\begin{aligned}
T_2 &= 2\alpha_2\rho^{-1}\,(w_1'u_1' - w_2'u_2') \\
&= 2\alpha_2\rho^{-1}\,[w_1'\,(u_1' - u_2') + u_2'(w_1' - w_2')],
\end{aligned}
$$

which implies $\|T_2\|_{L_2} \leq 2\alpha_2\rho^{-1}\,(\|w_1'\|_{L_\infty}\|u_1' - u_2'\|_{L_2} + \|u_2'\|_{L_\infty}\,\|w_1' - w_2'\|_{L_2})$. By virtue of the embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$, and since $u_i \in H_0^1 \cap H^2$, $w_i \in H^2$,

$w_i'(0) = 0$, $i = 1, 2$, it follows that

$$\begin{aligned}\|T_2\|_{L_2} &\le 2\alpha_2\rho^{-1}C_1C_\infty\,(\|u_1'' - u_2''\|_{L_2} + \|w_1'' - w_2''\|_{L_2})\\ &\le 2\alpha_2\rho^{-1}C_1C_\infty\left(\frac{\|z_1 - z_2\|_q}{\sqrt{\gamma}} + \frac{\|A(q)(z_1 - z_2)\|_q}{\sqrt{k/C_v}}\right)\\ &\le \frac{2\alpha_2C_1C_\infty}{\rho}\left(\frac{1}{\sqrt{\gamma}} + \sqrt{\frac{C_v}{k}}\right)|||z_1 - z_2|||_{A(q)}.\end{aligned} \tag{3.29b}$$

We have that

$$\begin{aligned}T_3 &= 2\alpha_2\rho^{-1}\left[\frac{\partial}{\partial x}L(\cdot, t_1)u_1' - \frac{\partial}{\partial x}L(\cdot, t_2)u_2'\right]\\ &= 2\alpha_2\rho^{-1}\left\{\frac{\partial}{\partial x}L(\cdot, t_1)(u_1' - u_2') + u_2'\left(\frac{\partial}{\partial x}L(\cdot, t_1) - \frac{\partial}{\partial x}L(\cdot, t_2)\right)\right\}\\ &= 2\alpha_2\rho^{-1}\{-2\pi\theta_\Gamma(t_1)\sin(2\pi\cdot)(u_1' - u_2') - 2\pi\sin(2\pi\cdot)u_2'(\theta_\Gamma(t_1) - \theta_\Gamma(t_2))\}.\end{aligned}$$

Applying Hypothesis (H2), the Sobolev embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ and the fact that $\|u_1' - u_2'\|_{L_2} \le \|u_1'' - u_2''\|_{L_2}$, we obtain the estimate

$$\begin{aligned}\|T_3\|_{L_2} \le{}& 4\alpha_2\rho^{-1}\pi\,\|\theta_\Gamma\|_{L_\infty(0,M_1)}\frac{|||z_1 - z_2|||_{A(q)}}{\sqrt{\gamma}}\\ &+ 4\alpha_2\rho^{-1}\pi C_1C_\infty K_{M_1}|t_1 - t_2|.\end{aligned} \tag{3.29c}$$

To obtain an estimate on $\|T_4\|_{L_2}$ we observe that

$$\begin{aligned}\|T_4\|_{L_2} &= 2\alpha_2\rho^{-1}\|w_1u_1'' - w_2u_2''\|_{L_2}\\ &= 2\alpha_2\rho^{-1}\|w_1(u_1'' - u_2'') + u_2''(w_1 - w_2)\|_{L_2}\\ &\le 2\alpha_2\rho^{-1}(\|w_1\|_{L_\infty}\|u_1'' - u_2''\|_{L_2} + \|u_2''\|_{L_2}\|w_1 - w_2\|_{L_\infty})\\ &\le 2\alpha_2\rho^{-1}\left(C_1C_\infty\frac{\|A(q)(z_1 - z_2)\|_q}{\sqrt{\gamma}} + \frac{\|z_2\|_q}{\sqrt{\gamma}}C_\infty\|w_1 - w_2\|_{H^1}\right)\\ &\le \frac{2\alpha_2C_\infty}{\rho\sqrt{\gamma}}(C_1|||z_1 - z_2|||_{A(q)} + |||z_2|||_{A(q)}\|w_1 - w_2\|_{H^1})\\ &\le \frac{2\alpha_2C_\infty}{\rho\sqrt{\gamma}}(C_1|||z_1 - z_2|||_{A(q)} + M_2(\|w_1 - w_2\|_{L_2} + \|w_1' - w_2'\|_{L_2})\}\\ &\le \frac{2\alpha_2C_\infty}{\rho\sqrt{\gamma}}\left(C_1|||z_1 - z_2|||_{A(q)} + M_2\frac{\|z_1 - z_2\|_q}{\sqrt{C_v/k}} + M_2\|w_1'' - w_2''\|_{L_2}\right)\end{aligned}$$

$$\leq \frac{2\alpha_2 C_\infty}{\rho\sqrt{\gamma}}\left(C_1 |||z_1 - z_2|||_{A(q)} + M_2\sqrt{\frac{k}{C_v}}\|z_1 - z_2\|_q + M_2\frac{\|A(q)(z_1 - z_2)\|_q}{\sqrt{k/C_v}}\right)$$

$$\leq \frac{2\alpha_2 C_\infty}{\rho\sqrt{\gamma}}\left(C_1 + M_2\sqrt{\frac{k}{C_v}} + M_2\sqrt{\frac{C_v}{k}}\right)|||z_1 - z_2|||_{A(q)}$$

$$= \frac{2\alpha_2 C_\infty}{\rho\sqrt{\gamma}}\left(C_1 + \frac{M_2(k + C_v)}{\sqrt{kC_v}}\right)|||z_1 - z_2|||_{A(q)}. \tag{3.29d}$$

By writing $T_5$ in the form

$$T_5 = 2\alpha_2\rho^{-1}[L(\cdot, t_1)u_1'' - L(\cdot, t_2)u_2'']$$
$$= 2\alpha_2\rho^{-1}[L(\cdot, t_1)(u_1'' - u_2'') + u_2''(L(\cdot, t_1) - L(\cdot, t_2))],$$

we immediately obtain the estimate

$$\|T_5\|_{L_2} \leq 2\alpha_2\rho^{-1}\left(\|\theta_\Gamma\|_{L_\infty(0,M_1)}\frac{\|z_1 - z_2\|_q}{\sqrt{\gamma}} + \|u_2''\|_{L_2}K_{M_1}|t_1 - t_2|\right)$$

$$\leq 2\alpha_2\rho^{-1}\left(\|\theta_\Gamma\|_{L_\infty(0,M_1)}\frac{|||z_1 - z_2|||_{A(q)}}{\sqrt{\gamma}} + \frac{\|z_2\|_q}{\sqrt{\gamma}}K_{M_1}|t_1 - t_2|\right) \tag{3.29e}$$

$$\leq \frac{2\alpha_2\|\theta_\Gamma\|_{L_\infty(0,M_1)}}{\rho\sqrt{\gamma}}|||z_1 - z_2|||_{A(q)} + \frac{2\alpha_2 M_2 K_{M_1}}{\rho\sqrt{\gamma}}|t_1 - t_2|.$$

The term $\|T_6\|_{L_2}$ satisfies the inequality

$$\|T_6\|_{L_2} = \|-2\alpha_2\rho^{-1}\theta_1(u_1'' - u_2'')\|_{L_2}$$

$$\leq \frac{2\alpha_2\theta_1}{\rho}\frac{\|z_1 - z_2\|_q}{\sqrt{\gamma}} \tag{3.29f}$$

$$\leq \frac{2\alpha_2\theta_1}{\rho\sqrt{\gamma}}|||z_1 - z_2|||_{A(q)}.$$

Observing that

$$T_7 = -12\alpha_4\rho^{-1}(u_1'^2u_1'' - u_2'^2u_2'')$$
$$= -12\alpha_4\rho^{-1}[u_1'^2(u_1'' - u_2'') + u_2''(u_1' + u_2')(u_1' - u_2')],$$

it follows that

$$\|T_7\|_{L_2} \leq 12\alpha_4\rho^{-1}(\|u_1'\|_{L_\infty}^2\|u_1'' - u_2''\|_{L_2} + \|u_2''\|_{L_2}\|u_1' + u_2'\|_{L_\infty}\|u_1' - u_2'\|_{L_\infty})$$
$$\leq 12\alpha_4\rho^{-1}(C_\infty^2 C_1^2\|u_1'' - u_2''\|_{L_2} + C_1(2C_1C_\infty)C_\infty\|u_1 - u_2\|_{H_1}).$$

Since $\|u\|_{H^1}^2 = \|u\|_{L_2}^2 + \|u'\|_{L_2}^2 \le 2\|u''\|_{L_2}^2$ and $u \in H_0^1 \cap H^2$, it follows that

$$
\begin{aligned}
\|T_7\|_{L_2} &\le \frac{12\alpha_4 C_1^2 C_\infty^2}{\rho} (\|u_1'' - u_2''\|_{L_2} + 2\sqrt{2}\|u_1'' - u_2''\|_{L_2}) \\
&= \frac{12\alpha_4 C_1^2 C_\infty^2}{\rho} (1 + 2\sqrt{2}) \|u_1'' - u_2''\|_{L_2} \qquad (3.29\text{g}) \\
&\le \frac{12\alpha_4 C_1^2 C_\infty^2}{\rho\sqrt{\gamma}} (1 + 2\sqrt{2}) |||z_1 - z_2|||_{A(q)}.
\end{aligned}
$$

Similarly, we use the identity

$$
\begin{aligned}
T_8 &= 30\alpha_6\rho^{-1}(u_1'^4 u_1'' - u_2'^4 u_2'') \\
&= 30\alpha_6\rho^{-1}[u_1'^4(u_1'' - u_2'') - u_2''(u_1'^4 - u_2'^4)] \\
&= 30\alpha_6\rho^{-1}[u_1'^4(u_1'' - u_2'') - u_2''(u_1'^3 + u_1'^2 u_2' + u_1' u_2'^2 + u_2'^3)(u_1' - u_2')]
\end{aligned}
$$

to obtain the estimate

$$
\begin{aligned}
\|T_8\|_{L_2} \le{}& 30\alpha_6\rho^{-1}\{\|u_1'\|_{L_\infty}^4\|u_1'' - u_2''\|_{L_2} \\
&+ \|u_2''\|_{L_2}\|u_1'^3 + u_1'^2 u_2' + u_1' u_2'^2 + u_2'^3\|_{L_\infty}\|u_1' - u_2'\|_{L_\infty}\} \\
\le{}& 30\alpha_6\rho^{-1}[C_1^4 C_\infty^4\|u_1'' - u_2''\|_{L_2} + C_1(4C_1^3 C_\infty^3)C_\infty\|u_1 - u_2\|_{H^1}] \\
\le{}& \frac{30\alpha_6 C_1^4 C_\infty^4}{\rho}(\|u_1'' - u_2''\|_{L_2} + 4\|u_1 - u_2\|_{H^1}) \qquad (3.29\text{h}) \\
\le{}& \frac{30\alpha_6 C_1^4 C_\infty^4}{\rho}(\|u_1'' - u_2''\|_{L_2} + 4\sqrt{2}\,\|u_1'' - u_2''\|_{L_2}) \\
\le{}& \frac{30\alpha_6 C_1^4 C_\infty^4}{\rho\sqrt{\gamma}}(1 + 4\sqrt{2})\,|||z_1 - z_2|||_{A(q)}.
\end{aligned}
$$

Substituting (3.29a)–(3.29h) into (3.28) we finally obtain

$$
\|f_3(q, t_1, z_1) - f_3(q, t_2, z_2)\|_{L_2} = \left\|\sum_{i=1}^{8} T_i\right\|_{L_2} \le C_6|t_1 - t_2| + C_7|||z_1 - z_2|||_{A(q)}, \tag{3.30}
$$

where $C_6 = \rho^{-1}(\|K_f\|_{L_2} + 4\alpha_2\pi C_1 C_\infty K_{M_1} + 2\alpha_2 M_2 K_{M_1}/\sqrt{\gamma})$, and

$$C_7 = \frac{1}{\rho\sqrt{\gamma}}\left\{4\alpha_2 C_1 C_\infty + 2\alpha_2\left[(1+2\pi)\|\theta_\Gamma\|_{L_\infty(0,M_1)} + \theta_1 + \frac{C_\infty M_2(k+C_v)}{\sqrt{kC_v}}\right]\right.$$
$$\left. + 12\alpha_4 C_1^2 C_\infty^2(1+2\sqrt{2}) + 30\alpha_6 C_1^4 C_\infty^4(1+4\sqrt{2})\right\} + \frac{2\alpha_2 C_1 C_\infty}{\rho}\sqrt{\frac{C_v}{k}}.$$

An analysis of the constants $C_i$, 1, 2, ..., 7, reveals that they are all bounded independently of $q$ on subsets of $\mathfrak{Q}$ of the form

$$\left\{(\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \in \mathfrak{Q} \;\middle|\; \begin{array}{c} 0 \le a_1 \le \rho \le b_1 < \infty, \quad 0 < a_2 \le C_v \le b_2 < \infty, \\ 0 < a_3 \le \gamma, \quad \beta \le b_3 < \infty, \quad \alpha_2 \le b_4 < \infty, \\ \alpha_4 \le b_5 < \infty, \quad \alpha_6 \le b_6 < \infty, \quad \theta_1 \le b_7 < \infty \end{array}\right\},$$

i.e., on any subset of $\mathfrak{Q}$ where $\rho$, $C_v$, $\beta$, $\alpha_2$, $\alpha_4$, $\alpha_6$, $\theta_1$ are bounded above and $\rho$, $C_v$, and $\gamma$ are bounded away from zero, and therefore on any compact subset of $\mathfrak{Q}$. This completes the proof of Theorem 3.10. ▌

We are now ready to state and prove the main local existence and uniqueness theorem for the solutions of the initial value problem (3.5).

THEOREM 3.11 (Local Existence and Uniqueness). *Let $q \in \mathfrak{Q}$ and $A(q)$:* dom$(A(q)) \subset Z_q \to Z_q$ *as defined in* (3.6), (3.7). *Then, for any initial data $z_0 \in$* dom$(A(q))$, *there exists $t_1 = t_1(z_0)$ such that the initial value problem* (3.5) *has a unique (strong) solution $z(t; q) \in C([0, t_1) : Z_q) \cap C^1((0, t_1) : Z_q)$.*

*Proof.* Let $z_0 \in \text{dom}(A(q))$ and let $U$ be a bounded open neighborhood of the origin in $\mathbb{R}_0^+ \times Z_{A(q)}$ containing the point $(0, z_0)$. By Theorem 3.10, the function $h_q : U \to Z_q$ defined by $h_q(t, z) \doteq F(q, t, z)$ satisfies the condition (F) on $U$. The local existence and uniqueness then follows from the fact that $A(q)$ is the infinitesimal generator of an analytic semigroup $T(t; q)$ on $Z_q$ with $0 \in \rho(A(q))$ (Theorem 3.7) and Theorem 3.9. ▌

## 4. CONTINUOUS DEPENDENCE ON THE PARAMETER $q$

In this section we will show that the semigroup $T(t; q)$ generated by $A(q)$ on $Z_q$ depends continuously on $q$, i.e., if $\{q^N\}_{N=1}^\infty \subset \mathfrak{Q}$ is a sequence of admissible parameters and $q^N \to q \in \mathfrak{Q}$ as $N \to \infty$, then the semigroup $T(t; q^N)$ generated by $A(q^N)$ on $Z_{q^N}$ "approaches" $T(t; q)$ in some sense.

The following is a version of the well known Trotter–Kato theorem due to Kurtz. Its proof can be found in [5, pp. 40–43].

THEOREM 4.1 (Trotter–Kato). *Let $A$ be the infinitesimal generator of a strongly continuous semigroup $T(t)$ on a Banach space $X$ with norm $\|\cdot\|$. Let $X^N$, $N = 1, 2, \ldots$, be a sequence of Banach spaces with norms $\|\cdot\|_N$, and $A^N$ the infinitesimal generator of a $C_0$-semigroup $T^N(t)$ on $X^N$. Let $\Pi^N \in \mathcal{L}(X, X^N)$, $N = 1, 2, \ldots$, be a sequence of bounded linear operators satisfying*

$$\lim_{N\to\infty} \|\Pi^N x\|_N = \|x\| \qquad \text{for all } x \in X. \tag{J1}$$

*Assume also that the following conditions hold*

(A) *(Stability) There exist constants $M_0 \geq 1$, $\omega_0 \in \mathbb{R}$ independent of $N$ such that*

$$\|T^N(t)\|_{\mathcal{L}(X^N)} \leq M_0 e^{\omega_0 t} \qquad \text{for all } t \geq 0 \text{ and } N = 1, 2, \ldots .$$

(B) *(Consistency) There exists a set $D \subset \operatorname{dom}(A)$ and a complex number $\lambda_0$ with $\operatorname{Re} \lambda_0 > \omega_0$ such that*

(i) $\Pi^N D \subset \operatorname{dom}(A^N)$ *for each* $N = 1, 2, \ldots,$

(ii) $\overline{(\lambda_0 - A)D} = X$, *and*

(iii) $\|A^N \Pi^N y - \Pi^N A y\|_N \to 0$ *as* $N \to \infty$ *for each* $y \in D$.

*Then for each $x \in X$, $\|T^N(t)\Pi^N x - \Pi^N T(t)x\|_N \to 0$ as $N \to \infty$ and the convergence is uniform on compact $t$-intervals.*

*Remarks.* (1) By using the uniform boundedness principle with minor modifications to take into account the varying spaces, it can be proved that condition (J1) implies $\|\Pi^N\|_{\mathcal{L}(X, X^N)} \leq C$ for some constant $C$ independent of $N$.

(2) Condition (B) implies that $\lambda_0 \in \rho(A) \bigcap_{N=1}^{\infty} \rho(A^N)$ and for each $x \in X$ there holds $\|R(\lambda; A^N)\Pi^N x - \Pi^N R(\lambda; A)x\|_N \to 0$. This is often referred to as the stability condition (see [5, p. 43]).

THEOREM 4.2 (Continuity of $T(t; q)$ with Respect to $q$). *Let $\{q^N = (\rho^N, C_v^N, \beta^N, \alpha_2^N, \alpha_4^N, \alpha_6^N, \theta_1^N, \gamma^N)\}_{N=1}^{\infty} \subset \mathfrak{Q}$ be a sequence of admissible parameters such that $q^N \to q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \in \mathfrak{Q}$ as $N \to \infty$. Let $T(t; q^N)$ and $T(t; q)$ be the analytic semigroups generated by $A(q^N)$ and $A(q)$ respectively. Then for each $z \in Z_q$ and $t \geq 0$, $\lim_{N\to\infty} \|T(t; q^N)z - T(t; q)z\|_{q^N} = 0$ and $\lim_{N\to\infty} \|T(t; q^N)z - T(t; q)z\|_{q^*} = 0$ for any $q^* \in \mathfrak{Q}$. Moreover, the convergence is uniform on compact $t$-intervals.*

*Proof.* Define $\Pi^N : Z_q \to Z_{q^N}$ to be the identity operator and let

$$z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q.$$

It follows that

$$\begin{aligned} \|\Pi^N z\|_{q^N}^2 &= \gamma^N \|u''\|_{L_2(\Omega)}^2 + \rho^N \|v\|_{L_2(\Omega)}^2 + \frac{C_v^N}{k} \|w\|_{L_2(\Omega)}^2 \\ &\xrightarrow{N\to\infty} \gamma \|u''\|_{L_2(\Omega)}^2 + \rho \|v\|_{L_2(\Omega)}^2 + \frac{C_v}{k} \|w\|_{L_2(\Omega)}^2 \\ &= \|z\|_q^2. \end{aligned}$$

Thus, condition (J1) of the Trotter–Kato theorem is satisfied. By Theorem 3.7, the semigroups $T(t; q^N)$ satisfy $\|T(t; q^N)\|_{\mathscr{L}(Z_{q^N})} \leq e^{-w(q^N)t}$. For the stability condition (A) of Theorem 4.1 we need this bound to be independent of $N$. This is achieved by choosing $M_0 = 1$ and $\omega_0 = 0$. Now, letting $D = \operatorname{dom}(A(q)) = \operatorname{dom}(A(q^N))$, condition (B.i) is trivial since $\Pi^N$ is the identity operator and (B.ii) holds for any choice of $\lambda_0 > 0$. Finally, for each

$$z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D$$

we have

$$\begin{aligned} &\|A(q^N)\Pi^N z - \Pi^N A(q) z\|_{q^N}^2 = \|A(q^N)z - A(q)z\|_{q^N}^2 \\ &= \left\| \begin{pmatrix} v \\ \beta^N v'' - \dfrac{\gamma^N}{\rho^N} u'''' \\ \dfrac{k}{C_v^N} w'' \end{pmatrix} - \begin{pmatrix} v \\ \beta v'' - \dfrac{\gamma}{\rho} u'''' \\ \dfrac{k}{C_v} w'' \end{pmatrix} \right\|_{q^N}^2 \\ &= \rho^N \left\| (\beta^N - \beta) v'' - \left(\frac{\gamma^N}{\rho^N} - \frac{\gamma}{\rho}\right) u'''' \right\|_{L_2(\Omega)}^2 + \frac{C_v^N}{k} \left\| k \left(\frac{1}{C_v^N} - \frac{1}{C_v}\right) w'' \right\|_{L_2(\Omega)}^2 \end{aligned}$$

$$\leq 2\rho^N \left[ (\beta^N - \beta)^2 \|v''\|^2_{L_2(\Omega)} + \left( \frac{\gamma^N}{\rho^N} - \frac{\gamma}{\rho} \right)^2 \|u''''\|^2_{L_2(\Omega)} \right]$$

$$+ kC_v^N \left( \frac{1}{C_v^N} - \frac{1}{C_v} \right)^2 \|w''\|^2_{L_2(\Omega)}$$

$$\to 0 \qquad \text{as } N \to \infty,$$

since $\beta^N \to \beta$, $\gamma^N \to \gamma$, $\rho^N \to \rho \neq 0$, and $C_v^N \to C_v \neq 0$ (note that $\rho \neq 0$ and $C_v \neq 0$ because $q \in \mathcal{Q}$). Thus, condition (B.iii) of the Trotter–Kato theorem, Theorem 4.1, is also satisfied. Hence, for each $z \in Z_q$ and $t \geq 0$

$$\lim_{N\to\infty} \|T(t; q^N)z - T(t; q)z\|_{q^N} = 0,$$

and the limit is uniform in $t$ on compact intervals.

The second part of the theorem follows immediately by noting that for any $q^* \in \mathcal{Q}$, $z \in Z_q$, and $t \geq 0$ one has

$$\|T(t; q^N)z - T(t; q)z\|^2_{q^*} \leq \max\left( \frac{\gamma^*}{\gamma^N}, \frac{\rho^*}{\rho^N}, \frac{C_v^*}{C_v^N} \right) \|T(t; q^N)z - T(t; q)z\|^2_{q^N},$$

$\gamma^N \to \gamma \neq 0$, $\rho^N \to \rho \neq 0$, and $C_v^N \to C_v \neq 0$, and therefore

$$\sup_{N=1,2,\ldots} \max\left( \frac{\gamma^*}{\gamma^N}, \frac{\rho^*}{\rho^N}, \frac{C_v^*}{C_v^N} \right) < \infty.$$

Hence, $\|T(t; q^N)z - T(t; q)z\|_{q^*} \to 0$ as $N \to \infty$ and the convergence is uniform on compact $t$-intervals. ∎

We note here that the continuity of the semigroup $T(t; q)$ with respect to the parameter $q$ does not imply the continuity of the solutions $z(t; q)$ with respect to $q$. A result stronger than the one in Theorem 3.10 is needed to achieve this goal. In fact, if $\delta < 1$ (i.e., if $F(q, t, z)$ satisfies the condition (F) with $\delta$ strictly less than 1), then one could use Gronwall's inequality together with the bound $\|(-A(q))^\delta\, T(t; q)z\|_q \leq Mt^{-\delta}\|z\|_q$ (see [35, Theorem 2.6.13]) to conclude that $\|z(t; q) - z(t; \hat{q})\|_{A^\delta(q)} \to 0$ as $q \to \hat{q}$. This is a very important issue since we are interested in parameter estimation. In particular, in order to establish the convergence of computational algorithms for parameter identification, one needs this continuous dependence.

## 5. Summary and Conclusions

In this paper we have developed an abstract framework for a one-dimensional dynamic mathematical model of structural phase transitions in shape memory alloys with thermodynamic potentials of the Landau–Ginzburg type. Using a state-space approach we transformed the system of partial differential equations that define the model into a semilinear Cauchy problem in an appropriate Hilbert space. We then showed that the differential operator $A(q)$ corresponding to the linear part of the system generates an exponentially stable analytic semigroup $T(t; q)$. We obtained spectral decompositions for $A(q)$ and $T(t; q)$ and explicit decay rates for $T(t; q)$ in terms of the physical constants of the model. We also showed that the non-linear part of the system is Lipschitz in the state variable with respect to the norm of the graph of $A(q)$, which leads to a proof of local existence of solutions. Finally, we also showed that the semigroup $T(t; q)$ depends continuously on the parameter $q$.

This approach provides a friendly mathematical framework suitable for further developments. In particular, this framework is appropriate for the development of computational algorithms for parameter identification and the study of the asymptotic behavior of the solutions. No results are yet known in these areas.

There is certainly much room for further study. For example, the proof of the well-posedness that we have given depends on the fact that $\gamma > 0$. For the case $\gamma = 0$, by leaving the term $2\alpha_2\theta_1 u_{xx}$ on the left-hand side of the Eq. (3.2a) we were able to show that the resulting linear operator $A(q)$ is quasidisipative and the generator of an analytic semigroup on the Hilbert space $H_0^1(\Omega) \times L_2(\Omega) \times L_2(\Omega)$ with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} \right\rangle \doteq 2\alpha_2\theta_1 \int_\Omega u'\hat{u}' + \rho \int_\Omega v\hat{v} + \frac{C_v}{k}\int_\Omega w\hat{w}.$$

However, we could not show that the nonlinear term is Lipschitz continuous in the state variable with respect to the norm of the operator $A(q)$. We plan to continue our efforts in this direction. Also, the results in this article depend strongly on the assumption $\beta > 0$. This approach does not work if $\beta = 0$.

The operator $A(q)$ given by (3.6), (3.7) is a fourth order differential operator in $u$ and second order in $v$ and $w$, while the nonlinear term

$$F(q, t, z), \qquad z = \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

given by (3.8)–(3.9b) is only second order in $u$ and first order in $v$ and $w$. For this reason, we strongly believe that this nonlinear term is Lipschitz in the state variable with respect to the graph-norm of the square root of $-A(q)$. If this conjecture were true, one could derive local existence for a much broader set of initial conditions, namely for $z_0 \in \mathrm{dom}([-A(q))^{1/2})$. The problem is that the square root of a differential operator is explicitly known only for a special subset of the natural boundary conditions and even in simple cases it is known only up to a multiplicative bounded operator [37]. Although the proof of the conjecture described above does not necessarily imply finding $[-A(q)]^{1/2}$ explicitly, it does involve finding bounds for $\|u''\|_{L_2}$, $\|v'\|_{L_2}$, and $\|w'\|_{L_2}$ in terms of

$$\left\| \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\|_{[-A(q)]^{1/2}} \qquad \text{for} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathrm{dom}([-A(q)]^{1/2}).$$

We also intent to devote efforts to solving this problem.

We have developed numerical approximations for the solutions of the system (3.2a), (3.2b). Details of the results in this area will be published in a forthcoming paper. Efforts are also underway to develop optimization algorithms for parameter estimation. This objective will involve the design of laboratory experiments to collect appropriate data for particular alloys. Estimation techniques will then be used to obtain accurate approximations to the model parameters.

Other areas of interest in which we plan to pursue further results are the inclusion of history-dependent stress-strain relations into the dynamic model and the study of the effects of viscosity and couple stress. Finally we note that most of the data for SMAs come from uniaxial tensile stretching experiments. For this reason, it is important from a practical point of view that the mathematical models include time-dependent stresses as possible boundary conditions. If $\gamma > 0$, an extra pair of boundary conditions is needed besides the stresses at both ends, and it is not clear what is appropriate in this case.

## Acknowledgments

The author thanks Professor John A. Burns for many valuable suggestions throughout this work. Acknowledgments are also given to an anonymous referee for several important suggestions and comments and for bringing reference [32] to the author's attention.

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